

New complex- and quaternion-hyperbolic reflection groups

Daniel Allcock*

1 June 1999

1991 mathematics subject classification: 22E40 (11F06, 11E39, 53C35)

Dedicated to my father John Allcock, 1940–1991

Abstract

We consider the automorphism groups of various Lorentzian lattices over the Eisenstein, Gaussian, and Hurwitz integers, and in some of them we find reflection groups of finite index. These provide new finite-covolume reflection groups acting on complex and quaternionic hyperbolic spaces. Specifically, we provide groups acting on $\mathbb{C}H^n$ for all $n < 6$ and $n = 7$, and on $\mathbb{H}H^n$ for $n = 1, 2, 3$, and 5 . We compare our groups to those discovered by Deligne and Mostow and show that our largest examples are new. For many of these Lorentzian lattices we show that the entire symmetry group is generated by reflections, and obtain a description of the group in terms of the combinatorics of a lower-dimensional positive-definite lattice. The techniques needed for our lower-dimensional examples are elementary, but to construct our best examples we also need certain facts about the Leech lattice. We give a new and geometric proof of the classifications of selfdual Eisenstein lattices of dimension ≤ 6 and of selfdual Hurwitz lattices of dimension ≤ 4 .

1. Introduction

In this paper we carry out complex and quaternionic analogues of some of Vinberg’s extensive study of reflection groups on real hyperbolic space. In [25] and [26] he investigated the symmetry groups of the integral quadratic forms $\text{diag}[-1, +1, \dots, +1]$, or equivalently the Lorentzian lattices $I_{n,1}$. He was able to describe these groups very concretely for $n \leq 17$, and extensions of his work by Vinberg and Kaplinskaja [27] and Borcherds [7] provide similar descriptions for all $n \leq 23$. In particular, the subgroup of $\text{Aut } I_{n,1}$ generated by reflections has finite index just when $n \leq 19$.

In this paper, we study the symmetry groups of Lorentzian lattices over the rings \mathcal{G} and \mathcal{E} of Gaussian and Eisenstein integers and the ring \mathcal{H} of Hurwitz integers (a discrete subring of the skew field \mathbb{H} of quaternions). Most of the paper is devoted to the most natural of such lattices, the selfdual ones. The symmetry groups of these lattices provide a large number of discrete groups generated by reflections and acting with finite-volume quotient on the hyperbolic spaces $\mathbb{C}H^n$ and $\mathbb{H}H^n$. We construct a total of 19 such groups, including groups acting on $\mathbb{C}H^7$ and $\mathbb{H}H^5$. At least one of our groups has been discovered before, in the work of Deligne and Mostow [18], Mostow [22] and Thurston [24], but our largest examples are new. To the author’s knowledge, quaternion-hyperbolic reflection groups not been studied before.

Our results and techniques have found important application in work of the author, J. Carlson and D. Toledo [2], [3] on the moduli space of complex cubic surfaces. Namely, this space is isomorphic to the Satake compactification of the quotient of $\mathbb{C}H^4$ by one of the reflection groups studied here. Furthermore, the moduli space of “marked” cubic surfaces may be realized as the Satake compactification of the quotient of $\mathbb{C}H^4$ by a congruence subgroup, which is also a reflection group in its own right.

The techniques used by Vinberg and others for the real hyperbolic case rely heavily on the fact that if a discrete group G is generated by reflections of $\mathbb{R}H^n$, then the mirrors of the reflections of G chop $\mathbb{R}H^n$ into pieces and each piece may be taken as a fundamental domain for G . Work with complex or quaternionic reflection groups is much more complicated, since hyperplanes have real codimension 2 or 4, and so the mirrors fail to chop hyperbolic space into pieces. Our solution to this problem is to avoid fundamental domains altogether. Each of our groups is defined as the subgroup $\text{Reflec } L$ of $\text{Aut } L$ generated by reflections, where L is a Lorentzian lattice over \mathcal{G} , \mathcal{E} or \mathcal{H} . (A Lorentzian lattice is a free module equipped with a Hermitian form of signature $- + \cdots +$.) Since $\text{Aut } L$ is an arithmetic group, to show that the quotient of $\mathbb{C}H^n$ or $\mathbb{H}H^n$ by $\text{Reflec } L$ has finite volume it suffices to show that $\text{Reflec } L$ has finite index in $\text{Aut } L$. In this case we say that L is reflective. Our basic strategy for proving a lattice L to be reflective is to prove first that $\text{Reflec } L$ acts with only finitely many orbits on the vectors of L of norm 0, and second that the stabilizer in $\text{Reflec } L$ of one such vector has finite index in the stabilizer in $\text{Aut } L$. We work mostly arithmetically,

* Supported by National Science Foundation Graduate and Postdoctoral Fellowships.

avoiding use of such tools as the bisectors introduced by Mostow for his study [21] of reflection groups on $\mathbb{C}H^2$.

However, there are certain steps in our constructions where geometric ideas play a key role. We express each of our Lorentzian lattices L in the form $\Lambda \oplus II_{1,1}$, where Λ is positive-definite and $II_{1,1}$ is a certain 2-dimensional lattice, the “hyperbolic plane”, with inner product matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It turns out that this description of L allows one to easily write down a large collection of reflections of L , parameterized by (a central extension of) the lattice Λ . It turns out that if Λ has enough vectors of norms 1 and 2, and provides a good covering of Euclidean space by balls, then one can automatically deduce that L is reflective. This implication is the content of Thm. 6.1. The rest of Section 6 is devoted to the application of this theorem and related ideas in the study of various examples. In particular, we prove that each of the selfdual Lorentzian lattices

$$I_{n,1}^{\mathcal{E}} \quad n = 1, 2, 3, 4, 7,$$

$$II_{n,1}^{\mathcal{G}} \quad n = 1, 5,$$

$$I_{n,1}^{\mathcal{H}} \quad n = 1, 2, 3, 5$$

is reflective. (These lattices are defined in Section 3 and characterized in Thm. 7.1.) For some of these, we obtain more detailed information. In particular, we prove that $\text{Reflec } I_{n,1}^{\mathcal{E}} = \text{Aut } I_{n,1}^{\mathcal{E}}$ for $n = 2, 3, 4$ or 7 and that $\text{Reflec } I_{n,1}^{\mathcal{H}}$ has index at most 4 in $\text{Aut } I_{n,1}^{\mathcal{H}}$ for $n = 2, 3$ or 5 . We also give explicit descriptions of the reflection groups of $I_{1,1}^{\mathcal{R}}$ and $II_{1,1}^{\mathcal{G}}$ as subgroups of certain Coxeter groups, acting on $\mathbb{C}H^1 \cong \mathbb{R}H^2$ and $\mathbb{H}H^1 \cong \mathbb{R}H^4$.

We note that the geometric ideas used here, namely that good coverings of Euclidean space lead to hyperbolic reflection groups, apply even when \mathbb{C} or \mathbb{H} is replaced by the nonassociative field \mathbb{O} of octaves (or octonions or Cayley numbers). In [4] we constructed two octave reflection groups acting on $\mathbb{O}H^2$ and one acting on $\mathbb{O}H^1 \cong \mathbb{R}H^8$, and interpreted these groups as the stabilizers of ‘lattices’ over a certain discrete subring of \mathbb{O} .

We provide background information on lattices in Section 2 and examples of them in Section 3; the latter should be referred to only as needed. Section 4 establishes our conventions regarding hyperbolic geometry. In Section 5 we relate certain geometric properties of a positive-definite lattice Λ to the reflection group of $\Lambda \oplus II_{1,1}$ and lay the foundations for Section 6, where we construct all of our examples. In Section 7 we explain the correspondence between primitive isotropic sublattices of $I_{n+1,1}$ and positive-definite selfdual lattices of dimension n . We use this correspondence to provide a quick geometric proof of the classification of selfdual lattices over \mathcal{E} and \mathcal{H} in dimensions ≤ 6 and ≤ 4 , respectively. The only examples besides the lattices \mathcal{E}^n and \mathcal{H}^n are the Coxeter-Todd lattice $\Lambda_6^{\mathcal{E}}$ and a quaternionic form $\Lambda_4^{\mathcal{H}}$ of the Barnes-Wall lattice. In Section 8 we show that our largest three groups, namely $\text{Reflec } L$ for $L = I_{7,1}^{\mathcal{E}}, I_{4,1}^{\mathcal{E}}$ and $II_{5,1}^{\mathcal{H}}$, are not among the 94 groups constructed in [18], [22] and [24]. We also sketch a proof that $\text{Reflec } I_{3,1}^{\mathcal{E}}$ does appear among these groups.

The easiest route to a new reflection group is our study of $II_{5,1}^{\mathcal{G}} = E_8^{\mathcal{G}} \oplus II_{1,1}^{\mathcal{G}}$, which acts on $\mathbb{C}H^5$. This requires only Lemmas 5.1 and 5.2 and Thm. 6.2. Most of our other examples require the more complicated Lemma 5.3 in place of 5.2. The arguments for $I_{7,1}^{\mathcal{E}}$ and $I_{5,1}^{\mathcal{H}}$ also require fairly involved space-covering arguments, involving embeddings of the Coxeter-Todd and Barnes-Wall lattices into the famous Leech lattice Λ_{24} . It is pleasing that Λ_{24} plays a role here, because our basic approach was inspired by Conway’s elegant description [10] of the isometry group of the \mathbb{Z} -lattice $II_{25,1} = \Lambda_{24} \oplus II_{1,1}$ in terms of the combinatorics of Λ_{24} . The embeddings of the Coxeter-Todd and Barnes-Wall lattices into Λ_{24} have also been used by Borchers [8] to produce interesting *real* hyperbolic reflection groups, acting on $\mathbb{R}H^{13}$ and $\mathbb{R}H^{17}$. Finally, the Leech lattice plays a much more direct role in [1], which constructs several other complex and quaternionic hyperbolic reflection groups, including one on $\mathbb{C}H^{13}$ and one on $\mathbb{H}H^7$.

Most of this paper is derived from the author’s Ph.D. thesis at Berkeley; he would like to thank his dissertation advisor, R. Borchers, for his interest and suggestions—in particular for suggesting that the quaternionic Barnes-Wall lattice would provide a reflection group on $\mathbb{H}H^5$.

2. Lattices

We denote by \mathcal{R} any one of the rings \mathcal{G} , \mathcal{E} , and \mathcal{H} —the Eisenstein, Gaussian, and Hurwitz integers. That is, $\mathcal{G} = \mathbb{Z}[i]$ and $\mathcal{E} = \mathbb{Z}[\omega]$, where $\omega = (-1 + \sqrt{-3})/2$ is a primitive cube root of unity. The ring \mathcal{H} is the integral span of its 24 units $\pm 1, \pm i, \pm j, \pm k$ and $(\pm 1 \pm i \pm j \pm k)/2$ in the skew field \mathbb{H} of quaternions. We write \mathbb{K} for the field $(\mathbb{C} \text{ or } \mathbb{H})$ naturally containing \mathcal{R} . Conjugation $x \mapsto \bar{x}$ denotes complex or quaternionic conjugation, as appropriate. For any element x of \mathbb{K} , we write $\operatorname{Re} x = (x + \bar{x})/2$ and $\operatorname{Im} x = (x - \bar{x})/2$ for the real and imaginary parts of x , and say that x is imaginary if $\operatorname{Re} x = 0$. If $X \subseteq \mathbb{K}$ then we write $\operatorname{Im} X$ for the set of imaginary elements of X . For any $x \in \mathbb{K}$, $x\bar{x}$ is a positive real number, and the absolute value $|x|$ of x is defined to be $(x\bar{x})^{1/2}$. It is convenient to define the element $\theta = \omega - \bar{\omega} = \sqrt{-3}$ of \mathcal{E} . We will sometimes also consider ω and θ as elements of \mathcal{H} , via the embedding $\mathcal{E} \rightarrow \mathcal{H}$ defined by $\omega \mapsto (-1 + i + j + k)/2$ or equally well by $\theta \mapsto i + j + k$.

A lattice Λ over \mathcal{R} is a free (right) module over \mathcal{R} equipped with a Hermitian form, which is to say a \mathbb{Z} -bilinear pairing (the inner product) $\langle \cdot | \cdot \rangle : \Lambda \times \Lambda \rightarrow \mathbb{K}$ such that

$$\langle x | y \rangle = \overline{\langle y | x \rangle} \quad \text{and} \quad \langle x | y\alpha \rangle = \langle x | y \rangle \alpha$$

for all $x, y \in \Lambda$ and $\alpha \in \mathcal{R}$. We use right modules and right-linear Hermitian forms so that lattice automorphisms can be described by matrices acting on the left. A Hermitian form on a (right) vector space over \mathbb{K} is defined similarly. Section 3 defines a number of interesting lattices and lists some of their properties. Sometimes we indicate that a lattice Λ is an \mathcal{R} -lattice by writing $\Lambda^{\mathcal{R}}$ or *somesuch*.

If $S \subseteq \Lambda$ then we denote by S^\perp its orthogonal complement: those elements of Λ whose inner products with all elements of S vanish. We say that Λ is nonsingular if $\Lambda^\perp = \{0\}$ and that Λ is integral if for all $x, y \in \Lambda$, the inner product $\langle x | y \rangle$ lies in \mathcal{R} . All lattices we consider will be integral and nonsingular unless otherwise specified. The dual Λ^* of Λ is the set of all \mathcal{R} -linear maps from Λ to \mathcal{R} . An integral lattice Λ is called selfdual if the natural map from Λ to Λ^* is onto. A selfdual lattice is sometimes called ‘unimodular’, because the matrix of inner products of any basis for Λ has determinant ± 1 ; we use ‘selfdual’ to avoid discussing determinants of quaternionic matrices.

The norm of a vector $v \in V$ is defined to be $v^2 = \langle v | v \rangle$; some authors call this the squared norm of v , but our convention is better for indefinite forms. We say that v is isotropic, or null, if $v^2 = 0$. A lattice is isotropic, or null, if each of its elements is. A lattice is called even if each of its elements has even norm and odd otherwise. A sublattice Λ' of Λ is called primitive if $\Lambda' = \Lambda \cap (\Lambda' \otimes \mathbb{Q})$. A vector v of Λ is called primitive if $v = w\alpha$ for $w \in \Lambda$ and $\alpha \in \mathcal{R}$ implies that α is a unit. Because the rings \mathcal{G} , \mathcal{E} and \mathcal{H} are principal ideal domains, a nonzero vector is primitive if and only if its \mathcal{R} -span is primitive as a sublattice. We will sometimes write $\langle v \rangle$ for the \mathcal{R} -span of $v \in \Lambda$.

We sometimes define an \mathcal{R} -lattice by describing a Hermitian form on \mathcal{R}^n . We do this by giving an $n \times n$ matrix (ϕ_{ij}) with entries in \mathcal{R} such that $\overline{\phi_{ij}} = \phi_{ji}$. Then the Hermitian form is given by

$$\langle (x_1, \dots, x_n) | (y_1, \dots, y_n) \rangle = \sum_{i,j=1}^n \bar{x}_i \phi_{ij} y_j.$$

We may also view a lattice as a subset of the vector space $V = \Lambda \otimes \mathbb{R}$ over the field $\mathbb{K} = \mathbb{R} \otimes \mathcal{R}$. The Hermitian form on Λ gives rise to one on V . If Λ is nonsingular then Λ^* may be identified with the set of vectors in V having \mathcal{R} -integral inner product with each element of Λ . Every nonsingular Hermitian form on a vector space V over \mathbb{K} is equivalent under $\operatorname{GL}(V)$ to one given by a diagonal matrix, with each diagonal entry being ± 1 . The signature of Φ is the ordered pair (n, m) where n (resp. m) is the number of $+1$ ’s (resp. -1 ’s). This characterizes Φ up to equivalence under $\operatorname{GL}(V)$. We write $\mathbb{K}^{n,m}$ for the vector space \mathbb{K}^{n+m} equipped with the standard Hermitian form of signature (n, m) ; the isometry group of $\mathbb{K}^{n,m}$ is the unitary group $U(n, m; \mathbb{K})$. The term “Lorentzian” is applied to various concepts in the study of real Minkowski space $\mathbb{R}^{n,1}$. By analogy with this we call a lattice Lorentzian if its signature is $(n, 1)$. Any isotropic sublattice of a Lorentzian lattice has dimension ≤ 1 .

If Λ is positive-definite then $\Lambda \otimes \mathbb{R}$ is a copy of Euclidean space under the metric $d(x, y) = \sqrt{(x - y)^2}$. Points of $\Lambda \otimes \mathbb{R}$ at maximal distance from Λ are called deep holes of Λ . The maximal distance is called the covering radius of Λ , because closed balls of that radius placed at lattice points exactly cover $\Lambda \otimes \mathbb{R}$. The lattice points nearest a deep hole are called the vertices of the hole. The covering radii of the \mathbb{Z} -lattices $\operatorname{Im} \mathcal{G}$,

$\text{Im } \mathcal{E}$ and $\text{Im } \mathcal{H}$ are $1/2$, $\sqrt{3}/2$ and $\sqrt{3}/2$, respectively. The first two are obvious and the last follows because $\text{Im } \mathcal{H}$ is the 3-dimensional lattice spanned by i , j and k . Any two deep holes of $\text{Im } \mathcal{R}$ are equivalent under translation by some element of $\text{Im } \mathcal{R}$.

Suppose that V is a Hermitian vector space over \mathbb{K} , $\xi \in \mathbb{K}$ is a root of unity and $v \in V$ has nonzero norm. We define the ξ -reflection in v to be the map

$$v \mapsto v - r(1 - \xi) \frac{\langle r | v \rangle}{r^2} . \quad (2.1)$$

This is an isometry of the right vector space V which fixes r^\perp pointwise and carries r to $r\xi$. (*Warning:* if $\mathbb{K} = \mathbb{H}$ then although the reflection acts by right scalar multiplication on r , it does not act this way on the entire \mathbb{H} -span of r . This is due to the noncommutativity of multiplication in \mathbb{H} .) Unless otherwise specified, we will use the term “reflection” to mean “reflection in a vector of positive norm”. Under the conventions of Section 4, (-1) -reflections in negative norm vectors act on hyperbolic space as inversions in points, rather than by reflections in hyperplanes. This is why we focus on positive-norm vectors. We call r^\perp the mirror of the reflection. Reflections of order 2, 3, \dots are sometimes called bifections, triflections, etc. A ξ -reflection is a bifection just if $\xi = -1$; in this case we recover the classical notion of a reflection.

Suppose L is an integral lattice. If $v \in L$ has norm 1 (resp. 2) then we say that v is a short (resp. long) root of L . Inspection of Eq. (2.1) reveals that if ξ is a unit of \mathcal{R} then ξ -reflection in any short root of L preserves L . Furthermore, bifections in long roots of L also preserve L . We define the reflection group $\text{Reflec } L$ to be the subgroup of $\text{Aut } L$ generated by reflections (in positive-norm vectors), and we say that L is reflective if $\text{Reflec } L$ has finite index in $\text{Aut } L$. In general, a group generated by reflections is called a reflection group. Since $\text{Aut } L$ is an arithmetic subgroup of the semisimple real Lie group $U(L \otimes \mathbb{R}; \mathbb{K})$, a theorem of Borel and Harish-Chandra [9] implies that it has finite covolume. It follows that L is reflective if and only if $\text{Reflec } L$ also has finite covolume. It may happen that $\text{Reflec } L$ contains reflections other than those in its roots, but we will not use them.

3. Reference: examples of lattices

This section contains background information on the various complex and quaternionic lattices we will use; it should be referred to only as necessary. We briefly define each lattice, list a few important properties and give references to the literature. The main source is [15, Chap. 4]. All lattices described here are integral. When lattices are described as subsets of \mathbb{K}^n it should be understood that the Hermitian form is $\langle (x_1, \dots, x_n) | (y_1, \dots, y_n) \rangle = \sum \bar{x}_i y_i$.

The simplest lattice is \mathcal{R}^n , which is obviously selfdual. Its symmetry group contains the left-multiplication by each diagonal matrix all of whose diagonal entries are units of \mathcal{R} . It is easy to see that the group of these coincides with the group generated by the reflections in the short roots. Adjoining to this group the permutations of coordinates, which are generated by bifections in long roots such as $(1, -1, 0, \dots, 0)$, we see that $\text{Aut } \mathcal{R}^n$ is a reflection group.

If Λ is a lattice then its real form is the \mathbb{Z} -module Λ equipped with the inner product $(x, y) = \text{Re} \langle x | y \rangle$. Here are three forms of the E_8 root lattice:

$$\begin{aligned} E_8 &= \frac{1}{2} \left\{ (x_1, \dots, x_8) \in \mathbb{Z}^8 \mid x_i \equiv x_j \pmod{2}, \sum x_i \equiv 0 \pmod{4} \right\} , \\ E_8^{\mathcal{G}} &= \frac{1}{1+i} \left\{ (x_1, \dots, x_4) \in \mathcal{G}^4 \mid x_i \equiv x_j \pmod{1+i}, \sum x_i \equiv 0 \pmod{2} \right\} , \\ E_8^{\mathcal{H}} &= \left\{ (x_1, x_2) \in \mathcal{H}^2 \mid x_1 + x_2 \equiv 0 \pmod{1+i} \right\} . \end{aligned}$$

It is straightforward to identify the real forms of these lattices with each other; each has covering radius 1 and minimal norm 2. Often the dimension of a lattice is indicated by a subscript. Unfortunately, this sometimes refers to its dimension as a \mathbb{Z} -lattice and sometimes to its dimension as an \mathcal{R} -lattice. There seems to be no universal solution to this notational problem.

Another set of useful even Gaussian lattices are

$$D_{2n}^{\mathcal{G}} = \left\{ (x_1, \dots, x_n) \in \mathcal{G}^n \mid \sum x_i \equiv 0 \pmod{1+i} \right\} ,$$

whose real forms are the D_{2n} root lattices. The D_4 lattice is also the real form of \mathcal{H} , scaled up by a factor of $2^{1/2}$. The covering radius of D_{2n} is $(n/2)^{1/2}$.

The Eisenstein lattice

$$D_3(\theta) = \{ (x, y, z) \in \mathcal{E}^3 \mid x + y + z \equiv 0 \pmod{\theta} \}$$

is one of the lattices $D_n(\sqrt{-3})$ introduced by Feit in [19]. It has 54 long roots and 72 vectors of norm 3; biflections in the former and triflections in the latter preserve the lattice. Its covering radius is 1; this can be seen as follows. According to [15, p. 126], the real form of the lattice $\{ (x, y, z) \in \mathcal{E}^3 \mid x \equiv y \equiv z \pmod{\theta} \}$ is the E_6 root lattice scaled up by $(3/2)^{1/2}$. This identification can be used to show that the real form of $D_3(\theta)$ is the real form of E_6^* scaled up by $(3/2)^{1/2}$, where E_6^* is the dual (over \mathbb{Z}) of E_6 . By [15, p. 127], the covering radius of E_6^* is $(2/3)^{1/2}$, so the covering radius of $D_3(\theta)$ is 1.

The Coxeter-Todd lattice $\Lambda_6^\mathcal{E}$ is a selfdual \mathcal{E} -lattice that is spanned by its long roots, which are also its minimal vectors. It is discussed at length in [14]; we quote just one of the definitions given there.

$$\Lambda_6^\mathcal{E} = \frac{1}{\theta} \left\{ (x_1, \dots, x_6) \in \mathcal{E} \mid x_i \equiv x_j \pmod{\theta}, \sum x_i \equiv 0 \pmod{3} \right\}.$$

Its automorphism group is the finite complex reflection group $6 \cdot U_4(3):2$, and $\Lambda_6^\mathcal{E}$ shares many interesting properties with E_8 and the Leech lattice Λ_{24} . We refer to [14] for details.

The quaternionic Barnes-Wall lattice is

$$\Lambda_4^\mathcal{H} = \frac{1}{1+i} \left\{ (x_1, \dots, x_4) \in \mathcal{H} \mid x_i \equiv x_j \pmod{(1+i)\mathcal{H}}, \sum x_i \in 2\mathcal{H} \right\}.$$

We may recognize the real form of $2^{1/2}\Lambda_4^\mathcal{H}$ by identifying the vector

$$(a_1 + b_1i + c_1j + d_1k, \dots, a_4 + b_4i + c_4j + d_4k)$$

with the vector in \mathbb{R}^{16} whose coordinates we arrange in the square array

$$\frac{4}{\sqrt{8}} \begin{array}{|c|c|c|c|} \hline a_1 & a_2 & a_3 & a_4 \\ d_1 & c_2 & d_3 & c_4 \\ \hline b_1 & d_2 & b_3 & d_4 \\ c_1 & b_2 & c_3 & b_4 \\ \hline \end{array}$$

where the inner product is the usual one on \mathbb{R}^{16} . This array may be taken to be (say) the left 4 columns of the 4×6 array in the MOG description [11, p. 97] of the Leech lattice Λ_{24} , and then the real form of $2^{1/2}\Lambda_4^\mathcal{H}$ is visibly the real Barnes-Wall lattice BW_{16} [15, Chap. 4].

Theorem 3.1. *The lattice $\Lambda_4^\mathcal{H}$ is selfdual and spanned by its minimal vectors, which have norm 2. Its automorphism group is generated by the biflections in its minimal vectors. Each class of $\Lambda_4^\mathcal{H}$ modulo $\Lambda_4^\mathcal{H}(1+i)$ is represented by a vector of norm at most 3. The deep holes of $\Lambda_4^\mathcal{H}$ coincide with the set $\{ \lambda(1+i)^{-1} \mid \lambda \in \Lambda_4^\mathcal{H}, \lambda^2 \equiv 1 \pmod{2} \}$.*

Proof: Proofs of all claims except the last appear in [5, Sect. 4.6]. Most of the rest of the work has been done for us by Conway and Sloane [13, Sect. 5]. They showed that the deep holes of BW_{16} nearest 0 are the halves of certain vectors $v \in BW_{16}$ of norm 12, and further that such v are not congruent modulo 2 to minimal vectors of BW_{16} . (They write Λ_{16} for BW_{16} .) After rescaling, we find that the deep holes of $\Lambda_4^\mathcal{H}$ nearest 0 are the halves of certain elements v of norm 6 in $\Lambda_4^\mathcal{H}$. Since each such v has even norm and is not congruent modulo $2 = -(1+i)^2$ to any root, it must map to 0 in $\Lambda_4^\mathcal{H}/\Lambda_4^\mathcal{H}(1+i)$. Therefore $v = \lambda(1+i)$ for some λ of norm 3 in $\Lambda_4^\mathcal{H}$ and so the deep holes nearest 0 have the form $v/2 = \lambda(1+i)/2 = (\lambda i)(1+i)^{-1}$.

The deep holes of $\Lambda_4^{\mathcal{H}}$ are the translates by lattice vectors of the deep holes nearest zero. That is, the set of deep holes coincides with the set

$$\{ \lambda(1+i)^{-1} \mid \lambda \in \Lambda_4^{\mathcal{H}} \text{ is congruent modulo } 1+i \text{ to a norm 3 lattice vector} \} .$$

The norms of any two lattice vectors that are congruent modulo $1+i$ have the same parity. Since each lattice vector is congruent to some vector of norm 0, 2 or 3, the set above coincides with the one in the statement of the theorem. \square

Now we describe some indefinite selfdual lattices. The lattice $I_{n,m}^{\mathcal{R}}$ is the \mathcal{R} -module \mathcal{R}^{n+m} equipped with the inner product given by the diagonal matrix

$$\text{diag}[+1, \dots, +1, -1, \dots, -1]$$

with n (resp. m) $+1$'s (resp. -1 's). The lattice $II_{1,1}^{\mathcal{R}}$ is the module \mathcal{R}^2 with inner product matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $\mathcal{R} = \mathcal{E}$ or \mathcal{H} then $II_{1,1}^{\mathcal{R}} \cong I_{1,1}^{\mathcal{R}}$ because one can find a norm 1 vector in the former lattice. If $\mathcal{R} = \mathcal{G}$ then $II_{1,1}$ is even, whereas $I_{1,1}$ is odd. We define the Gaussian lattices $II_{4m+n,n}^{\mathcal{G}}$ to be the lattices

$$II_{4m+n,n}^{\mathcal{G}} = E_8^{\mathcal{G}} \oplus \dots \oplus E_8^{\mathcal{G}} \oplus II_{1,1}^{\mathcal{G}} \oplus \dots \oplus II_{1,1}^{\mathcal{G}} ,$$

where there are m summands $E_8^{\mathcal{G}}$ and n summands $II_{1,1}^{\mathcal{G}}$. These lattices are even and selfdual. By Thm. 7.1, every indefinite selfdual lattice over \mathcal{R} appears among the examples just given. In particular, $\Lambda_6^{\mathcal{E}} \oplus II_{1,1}^{\mathcal{E}} \cong I_{7,1}^{\mathcal{E}}$ and $\Lambda_4^{\mathcal{H}} \oplus II_{1,1}^{\mathcal{H}} \cong I_{5,1}^{\mathcal{H}}$.

4. Hyperbolic space

The hyperbolic space $\mathbb{K}H^{n+1}$ ($n \geq 0$) is defined as the image in projective space $\mathbb{K}P^{n+1}$ of the set of vectors of negative norm in $\mathbb{K}^{n+1,1}$; its boundary $\partial\mathbb{K}H^{n+1}$ is the image of the (nonzero) null vectors. We write elements of $\mathbb{K}^{n+1,1}$ in the form $(\lambda; \mu, \nu)$ with $\lambda \in \mathbb{K}^{n,0}$ and $\mu, \nu \in \mathbb{K}$, with inner product

$$\langle (\lambda_1; \mu_1, \nu_1) | (\lambda_2; \mu_2, \nu_2) \rangle = \langle \lambda_1 | \lambda_2 \rangle + \bar{\mu}_1 \nu_2 + \bar{\nu}_1 \mu_2 .$$

This corresponds to a decomposition $\mathbb{K}^{n+1,1} \cong \mathbb{K}^{n,0} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We will often refer to points in projective space by naming vectors in the underlying vector space.

It is convenient to distinguish the isotropic vector $(0; 0, 1)$ and give it the name ρ . Every point of $\mathbb{K}H^{n+1} \cup \partial\mathbb{K}H^{n+1}$ except ρ has a unique preimage in $\mathbb{K}^{n+1,1}$ with inner product 1 with ρ , and so we may make the identifications

$$\begin{aligned} \mathbb{K}H^{n+1} &= \{(\lambda; 1, z) : \lambda \in \mathbb{K}^n, \lambda^2 + 2\text{Re}(z) > 0\} . \\ \partial\mathbb{K}H^{n+1} \setminus \{\rho\} &= \{(\lambda; 1, z) : \lambda \in \mathbb{K}^n, \lambda^2 + 2\text{Re}(z) = 0\} . \end{aligned} \tag{4.1}$$

We define the height of a vector $v \in \mathbb{K}^{n+1,1}$ to be $\text{ht } v = \langle \rho | v \rangle$. For $v = (\lambda; \mu, \nu)$ this is simply μ . For vectors of any fixed norm, the height function measures how far away from ρ the corresponding points in projective space are; the smaller the height, the closer to ρ . We will sometimes say that a vector v' has height less than that of another vector v . By this we will mean $|\text{ht } v'| < |\text{ht } v|$.

We say that the vector $(\lambda; \mu, \nu)$ of height $\mu \neq 0$ lies over $\lambda\mu^{-1} \in \mathbb{K}^n$. It is obvious that all the scalar multiples of any given vector of nonzero height lie over the same point of \mathbb{K}^n , so we may think of points in projective space (except for those in ρ^\perp) as lying over elements of \mathbb{K}^n . The geometric content of this definition is that the lines in $\mathbb{K}P^{n+1}$ passing through ρ and meeting $\mathbb{K}H^{n+1}$ are in one-to-one correspondence with \mathbb{K}^n . The points in the line associated to $\lambda \in \mathbb{K}^n$ are the scalar multiples of those of the form $(\lambda; 1, z)$ with $z \in \mathbb{K}$, which are precisely the points of $\mathbb{K}P^{n+1}$ lying over λ . We gave two special cases in Eq. (4.1). In particular, the family of height one isotropic vectors lying over λ is parameterized by the elements of $\text{Im } \mathbb{K}$. This description of $\partial\mathbb{K}H^{n+1} \setminus \{\rho\}$ as a bundle over \mathbb{K}^n with fiber $\text{Im } \mathbb{K}$ will help us relate the properties of lattices in \mathbb{K}^n to properties of groups acting on $\mathbb{K}H^{n+1}$.

The subgroup of $U(n+1, 1; \mathbb{K})$ fixing ρ contains transformations $T_{x,z}$ (with $x \in \mathbb{K}^n$, $z \in \text{Im } \mathbb{K}$) defined by

$$\begin{aligned} \rho &\mapsto \rho \\ T_{x,z}: \quad (0; 1, 0) &\mapsto (x; 1, z - x^2/2) \\ (\lambda; 0, 0) &\mapsto (\lambda; 0, -\langle x|\lambda \rangle) \quad \text{for each } \lambda \in \mathbb{K}^n. \end{aligned} \tag{4.2}$$

(The map is defined in terms of some unspecified but fixed inner product on \mathbb{K}^n .) We call these maps translations. If we regard elements of $\mathbb{K}^{n+1,1}$ as column vectors then $T_{x,z}$ acts by multiplication on the left by the matrix

$$\begin{pmatrix} I_n & x & 0 \\ 0 & 1 & 0 \\ -x^* & z - x^2/2 & 1 \end{pmatrix}.$$

We have written x^* for the linear function $y \mapsto \langle x|y \rangle$ on $\mathbb{K}^{n,0}$ defined by x . We have the relations

$$T_{x,z} \circ T_{x',z'} = T_{x+x', z+z'+\text{Im}\langle x'|x \rangle} \tag{4.3}$$

$$T_{x,z}^{-1} = T_{-x, -z} \tag{4.4}$$

$$T_{x,z}^{-1} \circ T_{x',z'}^{-1} \circ T_{x,z} \circ T_{x',z'} = T_{0, 2\text{Im}\langle x'|x \rangle}, \tag{4.5}$$

which are most easily verified in the order listed. These relations make it clear that the translations form a group and that its center and commutator subgroup coincide and consist of the $T_{0,z}$. We call elements of this subgroup central translations. The translations form a (complex or quaternionic) Heisenberg group which acts freely and transitively on $\partial\mathbb{K}H^{n+1} \setminus \{\rho\}$. If $v \in \mathbb{K}^{n+1,1}$ lies over $\lambda \in \mathbb{K}^n$ then $T_{x,z}(v)$ lies over $\lambda + x$. That is, the translations act in the natural way (by translations) on the points of \mathbb{K}^n over which vectors in $\mathbb{K}^{n+1,1}$ lie.

We note that these constructions all make sense when $\mathbb{K} = \mathbb{R}$, and even simplify. Since $\text{Im } \mathbb{R} = 0$, the translations form an abelian group, which is just the obvious set of translations in the usual upper half-space model for $\mathbb{R}H^{n+1}$. The obvious projection map from the upper half-space to \mathbb{R}^n carries points of $\mathbb{R}H^{n+1}$ to the points of \mathbb{R}^n over which they lie, in the sense defined above. This is the source of the terminology.

The simultaneous stabilizer of $(0; 1, 0)$ and $(0; 0, 1)$ is the unitary group $U(n, 0; \mathbb{K})$, which fixes pointwise the second summand of the decomposition $\mathbb{K}^{n+1,1} = \mathbb{K}^{n,0} \oplus \mathbb{K}^{1,1}$. If S is an element of this unitary group then matrix computations reveal

$$S \circ T_{x,z} \circ S^{-1} = T_{Sx, z}. \tag{4.6}$$

5. Reflections in Lorentzian lattices

The Lorentzian lattices we will consider all have the form $\Lambda \oplus II_{1,1}$, where Λ is a positive-definite \mathcal{R} -lattice and $II_{1,1}$ is the 2-dimensional selfdual lattice defined by the matrix $II_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In general we will write L for a Lorentzian lattice $\Lambda \oplus II_{1,1}$, where Λ and even \mathcal{R} may be left unspecified, except that Λ will always be positive-definite. We write elements of $L = \Lambda \oplus II_{1,1}$ in the form $(\lambda; \mu, \nu)$ with $\lambda \in \Lambda$ and $\mu, \nu \in \mathcal{R}$. This embeds L in the description of $\mathbb{K}^{n+1,1}$ given in Section 4 and allows us to transfer to L several important concepts defined there. In particular, $\rho = (0; 0, 1)$ is an element of L and we define the height of elements of L as before. For $v \in L$ of nonzero height, the point of $\Lambda \otimes \mathbb{R}$ over which v lies is in $\Lambda \otimes \mathbb{Q}$ but not necessary in Λ .

There are two basic ideas in this section. First, this description of L provides a way to write down a large collection of reflections of L , essentially parameterized by the elements of a discrete Heisenberg group of translations. The second idea is that if r is a root of L and v is a null vector in $\mathbb{K}^{n+1,1}$, and if r and v lie over points of \mathbb{K}^n that are sufficiently close, then by applying a reflection of L one can reduce the height of v . (This reflection might be in some root other than r .)

Both of these ideas can be found in the simpler setting of real hyperbolic space, in Conway's study [10] of the automorphism group of the Lorentzian \mathbb{Z} -lattice $II_{25,1} = \Lambda_{24} \oplus II_{1,1}$. Here Λ_{24} is the Leech lattice, and Conway found a set of reflections permuted freely by a group of translations naturally isomorphic to

the additive group of Λ_{24} . By using facts about the covering radius of Λ_{24} together with the second idea described above, he was able to prove that these reflections generate the entire reflection group of $II_{25,1}$.

The major complication in transferring this approach to our setting is that the discrete group of translations is no longer a copy of Λ but a central extension of Λ by $\text{Im } \mathcal{R}$. This issue dramatically complicates the precise formulation (Table 5.1) of the second main idea. For example, it is complicated to state exactly what happens when one can't *quite* reduce the height of $v \in \mathbb{K}^{n+1,1}$ by using a reflection.

We begin by finding the translations in $\text{Aut } L$ and showing that under simple conditions, $\text{Reflec } L$ contains a large number of them. The translation $T_{x,z}$ preserves L just if $x \in \Lambda$ and $z - x^2/2 \in \mathcal{R}$. If $\mathcal{R} = \mathcal{E}$ or \mathcal{H} then for any given $x \in \Lambda$ we may choose $z \in \text{Im } \mathbb{K}$ such that $T_{x,z} \in \text{Aut } L$, by taking $z = 0$ or $\theta/2$ according as x^2 is even or odd. If $\mathcal{R} = \mathcal{G}$ then such a z exists if and only if x^2 is even; z may then be taken to be zero. The different rings behave differently because \mathcal{E} and \mathcal{H} contain elements with half-integral real parts, while \mathcal{G} does not. All the central translations $T_{0,z}$ with $z \in \mathcal{R}$ lie in $\text{Aut } L$ —they fix Λ pointwise and act by isometries of $II_{1,1}$. The assertions of the next lemma are precise formulations of the idea that if $\text{Aut } \Lambda$ contains many reflections then $\text{Reflec } L$ contains many translations.

Lemma 5.1. *Let $L = \Lambda \oplus II_{1,1}$ for some positive-definite \mathcal{R} -lattice Λ . Define*

$$\Lambda_0 = \{ x \in \Lambda \mid T_{x,z} \in \text{Reflec } L \text{ for some } z \in \text{Im } \mathbb{K} \} \text{ and } \\ \mathcal{S} = \{ z \in \text{Im } \mathcal{R} \mid T_{0,z} \in \text{Reflec } L \} .$$

- (i) *If $\mathcal{R} = \mathcal{E}$ or \mathcal{H} then Λ_0 contains the short roots of Λ .*
- (ii) *If r is a long root of Λ then $2r \in \Lambda_0$. Furthermore, if r has inner product 1 with some element of Λ then r itself lies in Λ_0 .*
- (iii) *\mathcal{S} contains the integral span of the elements $2\text{Im}\langle x|y \rangle$ with $x, y \in \Lambda_0$.*
- (iv) *If the roots of $\Lambda \neq \{0\}$ span Λ up to finite index then the stabilizer of ρ in $\text{Reflec } L$ has finite index in the stabilizer in $\text{Aut } L$.*

Proof: Let R be a ξ -reflection of Λ with mirror M . We regard R as acting on L , fixing the summand $II_{1,1}$ pointwise. If $T_{x,z} \in \text{Aut } L$ then $T_{x,z}^{-1} \circ R \circ T_{x,z} \in \text{Reflec } L$. By Eqs. (4.4), (4.6) and (4.3),

$$T_{x,z}^{-1} \circ R \circ T_{x,z} \circ R^{-1} = T_{-x,-z} \circ T_{Rx,z} = T_{Rx-x, -\text{Im}\langle Rx|x \rangle} ,$$

proving that

$$Rx - x \in \Lambda_0 \tag{5.1}$$

for all $x \in \Lambda$ and all reflections R of Λ . The geometric picture behind this computation is that both M and its translate by $T_{x,z}^{-1}$ pass through ρ and are parallel there; we have constructed a translation out of reflections in two parallel mirrors.

(i) If r is a short root of Λ then we let $x = r\omega$ and R be the $(-\omega)$ -reflection in r . Then Λ_0 contains $Rx - x = r(-\omega)\omega - r\omega = r(-\bar{\omega} - \omega) = r$.

(ii) If r is a long root of Λ then we let $x = -r$, let R be the biflecction in r , and observe $Rx - x = 2r$. To prove the second claim, suppose $x \in \Lambda$ has inner product -1 with r and take R to be the biflecction in r . Then $Rx - x \in \Lambda_0$ is proportional to r and has inner product 2 with r , so it coincides with r .

(iii) Follows immediately from Eq. (4.5) by taking commutators of translations of $\text{Reflec } L$.

(iv) The null vectors of height 1 in L are exactly those vectors $(\lambda; 1, z)$ with $\lambda \in \Lambda$, $z \in \mathcal{R}$ and $\text{Re } z = -\lambda^2/2$, and the translations in $\text{Aut } L$ permute them transitively. Since the simultaneous stabilizer of ρ and one of these, say $(0; 1, 0)$, is the finite group $\text{Aut } \Lambda$, it suffices to prove that the group of translations in $\text{Reflec } L$ has finite index in the group of those in $\text{Aut } L$. This follows from (i)–(iii): Λ_0 has finite index in Λ and \mathcal{S} has finite index in $\text{Im } \mathcal{R}$. \square

Now we will exhibit a large number of reflections of L . It is straightforward to enumerate the roots of L of any given height h ; For $h = 1$ one finds that these are the vectors

$$\begin{array}{ll} \text{Norm 2:} & (\lambda; 1, z), \quad \text{Re } z = (2 - \lambda^2)/2 \\ \text{Norm 1:} & (\lambda; 1, z), \quad \text{Re } z = (1 - \lambda^2)/2 , \end{array}$$

with $\lambda \in \Lambda$ and $z \in \mathcal{R}$. If $\mathcal{R} = \mathcal{E}$ or \mathcal{H} , then height 1 roots of both norms lie over each $\lambda \in \Lambda$, and the translations of L act simply transitively on each set. If $\mathcal{R} = \mathcal{G}$ then height one roots lie over each $\lambda \in \Lambda$: long roots over λ of even norm and short roots over λ of odd norm. Again, the translations act simply transitively on each set. The differing behavior of the different rings is another manifestation of the fact that \mathcal{E} and \mathcal{H} have elements with half-integer real part, while \mathcal{G} does not. One may also enumerate roots of larger heights—for example, if Λ is an \mathcal{E} -lattice, then there are short roots of L of height θ over each $\lambda\theta^{-1} \in \Lambda\theta^{-1}$ with $\lambda^2 \equiv 1$ modulo 3.

Now we will discuss the second idea of this section: the effects of reflections in roots of small height h . This will occupy the rest of the section.

Lemma 5.2. *Suppose Λ is a positive-definite \mathcal{G} -lattice and $L = \Lambda \oplus II_{1,1}$. Suppose r is a long root of L of height 1 lying over $\lambda \in \Lambda$, and that v is an isotropic vector of $L \otimes \mathbb{R}$ of height 1 that lies over $\ell \in \Lambda \otimes \mathbb{R}$. Suppose that $(\ell - \lambda)^2 < \sqrt{3}$. Then there is another long root $r' \in L$ of height 1, also lying over λ , such that the biflection in r' reduces the height of v .*

Proof: Since v has height 1 and norm 0 and lies over ℓ , we know that for some $w \in \text{Im } \mathbb{K}$ we have $v = (\ell; 1, w - \ell^2/2)$. Similarly, we deduce that

$$r = \left(\lambda; 1, z_0 + \frac{2 - \lambda^2}{2} \right) \quad (5.2)$$

for some $z_0 \in \text{Im } \mathbb{K}$. Every other height 1 long root of L lying over λ has the form $r' = r + (0; 0, z)$ for some $z \in \text{Im } \mathcal{G}$. We will obtain the theorem by choosing z appropriately.

We have

$$\langle r'|v \rangle = \langle \lambda|\ell \rangle + (w - \ell^2/2) + (\bar{z}_0 + \bar{z} + 1 - \lambda^2/2) \quad (5.3)$$

$$= 1 - \frac{1}{2} (\ell^2 - 2\langle \lambda|\ell \rangle + \lambda^2) + w + \bar{z}_0 + \bar{z} \quad (5.4)$$

$$= 1 - \frac{1}{2} (\ell^2 - \langle \lambda|\ell \rangle - \langle \ell|\lambda \rangle + \lambda^2) + \frac{1}{2} (\langle \lambda|\ell \rangle - \langle \ell|\lambda \rangle) + w + \bar{z}_0 + \bar{z} \quad (5.5)$$

$$= \left[1 + \frac{1}{2} (\ell - \lambda)^2 \right] + [\text{Im} \langle \lambda|\ell \rangle + w + \bar{z}_0 + \bar{z}] \quad (5.6)$$

$$= a + B, \quad (5.7)$$

where a is the first bracketed expression and B is the second. The important thing to observe here is that a depends on $(\ell - \lambda)^2$, which is bounded by hypothesis, and B depends on z , over which we have some control. Let v' be the image of v under biflection in r' . Since $v' = v - r'\langle r'|v \rangle$, we have

$$\begin{aligned} \langle \rho|v' \rangle &= \langle \rho|v \rangle - \langle \rho|r' \rangle \langle r'|v \rangle \\ &= 1 - (1)(a + B) \\ &= (\ell - \lambda)^2/2 - B. \end{aligned}$$

Since the covering radius of $\text{Im } \mathcal{G}$ is $1/2$, we may choose z so that $|B| \leq 1/2$. Then

$$|\langle \rho|v' \rangle|^2 = \left| \frac{(\ell - \lambda)^2}{2} \right| + |B|^2 < \left| \frac{\sqrt{3}}{2} \right|^2 + \left| \frac{1}{2} \right|^2 = 1,$$

so that $\text{ht } v' < \text{ht } v$. □

Lemma 5.3. *Suppose Λ is a positive-definite \mathcal{R} -lattice and $L = \Lambda \oplus II_{1,1}$. Let $h = 1$ if $\mathcal{R} = \mathcal{G}$, $h = 1$ or θ if $\mathcal{R} = \mathcal{E}$, and $h = 1$ or $1 + i$ if $\mathcal{R} = \mathcal{H}$. Suppose r is a short root of L of height h lying over λh^{-1} , with $\lambda \in \Lambda$. Let $v \in L \otimes \mathbb{R}$ be isotropic, have height 1, and lie over $\ell \in \Lambda \otimes \mathbb{R}$. Set $D^2 = (\ell - \lambda h^{-1})^2$ and suppose $D^2 \leq 1/|h|^2$. Then there exists a short root r' of L , also of height h and lying over λh^{-1} , such that one of the following holds:*

- (i) some reflection in r' carries v to a vector of smaller height than v ;
- (ii) $D^2 = 1/|h|^2$ and $\langle r'|v \rangle = 0$; or
- (iii) $\mathcal{R} = \mathcal{H}$, $h = 1 + i$, $D^2 = 1/|h|^2 = 1/2$ and $\langle r'|v \rangle = (1 + i)/2$.

Proof: From the given norms and heights of v and r , together with the fact that they lie over ℓ and λh^{-1} , we deduce

$$v = (\ell; 1, w - \ell^2/2) \quad \text{and} \quad r = \left(\lambda; h, z_0 + \frac{1 - \lambda^2}{2|h|^2} h \right) \quad (5.8)$$

for some $w \in \text{Im } \mathbb{K}$ and $z_0 \in \mathbb{K}$ such that $\text{Re}(\bar{h}z_0) = 0$. The other height h short roots of L lying over λh^{-1} have the form $r' = r + (0; 0, z)$ for $z \in \mathcal{R}$ such that $\text{Re}(\bar{h}z) = 0$. The basic idea is similar to that of Lemma 5.2: we will try to choose z , together with a unit ξ of \mathcal{R} , such that the ξ -reflection in r' carries v to a vector of smaller height. It may happen that no such choice is possible, which leads to the cases (ii) and (iii) of the theorem.

A calculation similar to Eqs. (5.3)–(5.7) reveals that

$$\begin{aligned} \langle r'|v \rangle &= \langle \lambda | \ell \rangle + \bar{h} \left(w - \frac{\ell^2}{2} \right) + \overline{\left(z_0 + z + \frac{(1 - \lambda^2)h}{2|h|^2} \right)} \\ &= h^{-1} \left[\left(\frac{1}{2} - \frac{|h|^2}{2} D^2 \right) + (|h|^2 \text{Im} \langle \lambda h^{-1} | \ell \rangle + |h|^2 w + h \bar{z}_0 + h \bar{z}) \right] \\ &= h^{-1} [a + B] \end{aligned} \quad (5.9)$$

where $a = (1 - |h|^2 D^2)/2$ is the real part of the term in brackets and B is the imaginary part. The slight difference between the terms a in Eqs. (5.7) and (5.9) is due to the replacement of $2 - \lambda^2$ in Eq. (5.2) by $1 - \lambda^2$ in Eq. (5.8), which is due to the fact that r is now a short root.

We take v' to be the image of v under ξ -reflection in r' (we will choose ξ later). Since $v' = v - r'(1 - \xi)\langle r'|v \rangle$, we have

$$\begin{aligned} \langle \rho | v' \rangle &= \langle \rho | v \rangle - \langle \rho | r' \rangle (1 - \xi) \langle r' | v \rangle \\ &= 1 + \frac{h(\xi - 1)\bar{h}}{|h|^2} [a + B]. \end{aligned} \quad (5.10)$$

By hypothesis, $D^2 \leq 1/|h|^2$, so $a \in [0, 1/2]$. We may change the value of B by $h\bar{z}$, where z may be any element of $\mathcal{R} \cap \text{Im}(\bar{h}\mathbb{K})$. That is, we may change B by any element of

$$\begin{aligned} h \cdot \overline{\mathcal{R} \cap (h \cdot \text{Im } \mathbb{K})} &= h \cdot (\mathcal{R} \cap (\text{Im } \mathbb{K}) \cdot \bar{h}) \\ &= (h\mathcal{R}) \cap h \cdot (\text{Im } \mathbb{K}) \cdot \bar{h} \\ &= (h\mathcal{R}) \cap \text{Im } \mathbb{K} \\ &= \text{Im}(h\mathcal{R}). \end{aligned}$$

We will try to choose ξ and z so that (5.10) has absolute value less than $1 = \langle \rho | v \rangle$. This requires treating the different possibilities for \mathcal{R} and h separately. We will treat only the case $\mathcal{R} = \mathcal{H}$, $h = 1 + i$, which is more involved than the other four cases.

We write B as $bi + cj + dk$ with $b, c, d \in \mathbb{R}$. We first carry out a computation that will allow us to use the 24 units of \mathcal{H} effectively: we claim that there is a unit ξ' of \mathcal{H} with $\text{Re } \xi' = -1/2$ such that

$$|1 + \xi'(a + B)|^2 = (a - 1/2)^2 + (|b| - 1/2)^2 + (|c| - 1/2)^2 + (|d| - 1/2)^2. \quad (5.11)$$

For any unit ξ' , the left side is just the square of the distance between $a + B$ and $-\bar{\xi}'$ (proof: left-multiply by $1 = |-\bar{\xi}'|^2$). Setting $-\bar{\xi}' = (1 \pm i \pm j \pm k)/2$, with each of its i, j and k components having the same sign as the corresponding component of B (or a random sign if that component vanishes), the right hand side becomes another expression for this squared distance, proving the claim.

Next, we investigate our freedom to choose B . By choice of z we may vary B by any element of $\text{Im}(h\mathcal{H})$. It is easy to check that

$$\text{Im}((1+i)\mathcal{H}) = \{ bi + cj + dk \mid b, c, d \in \mathbb{Z}, b + c + d \equiv 0 \pmod{2} \}.$$

That is, $\text{Im}(h\mathcal{H})$ is spanned by $j+k$, $k+i$ and $i-k$, so by choice of z we may take $b \in (-1, 1]$ and $c, d \in [0, 1)$.

Suppose for a moment that there is a unit ξ of \mathcal{H} such that

$$\xi' = \frac{h(\xi - 1)\bar{h}}{|h|^2}, \quad (5.12)$$

where ξ' is as in Eq. 5.11. Then by Eqs. (5.10), (5.12) and (5.11),

$$\begin{aligned} |\text{ht } v'|^2 &= |1 + \xi'(a + B)|^2 \\ &= (a - 1/2)^2 + (|b| - 1/2)^2 + (|c| - 1/2)^2 + (|d| - 1/2)^2. \end{aligned} \quad (5.13)$$

We have already shown that $a \in [0, 1/2]$. By this and the constraints on b, c and d obtained above, we see that the right hand side of Eq. (5.13) is less than $1 = |\text{ht } v|^2$ (so that conclusion (i) applies) unless $a = 0$, $b \in \{0, 1\}$ and $c = d = 0$. In this exceptional case, $a = 0$ implies that $D^2 = 1/|h|^2$, and $\langle r'|v \rangle$ can be read from Eq. (5.9). If $b = 0$ we have $r' \perp v$ and conclusion (ii) applies, and if $b = 1$ then conclusion (iii) applies.

It remains only to show that given a unit ξ' of \mathcal{H} with $\text{Re } \xi' = -1/2$, there is another unit ξ of \mathcal{H} satisfying Eq. (5.12). We simply solve for ξ :

$$\xi = |h|^2 \cdot h^{-1} \xi' \bar{h}^{-1} + 1 = \frac{(1-i)\xi'(1+i)}{\sqrt{2}} + 1. \quad (5.14)$$

The most straightforward way to show that ξ is a unit of \mathcal{H} is to simply evaluate the right hand side of Eq. (5.14) for each of the eight possibilities $\xi' = (-1 \pm i \pm j \pm k)/2$. (What is really going on here is that the units of \mathcal{H} together with $(1+i)/\sqrt{2}$ generate the binary octahedral group, which normalizes the binary tetrahedral group consisting of the units of \mathcal{H} .) \square

The idea used in the proofs of the last two lemmas can also be applied for some other values of h . The cases stated above are the ones that will be used later, but for completeness we summarize in table 5.1 all the cases we have been able to treat with this method. The table should be read as follows. Suppose Λ is an \mathcal{R} -lattice, $L = \Lambda \oplus II_{1,1}$, and r is a short root of L whose height h appears in the table, lying over λh^{-1} , with $\lambda \in \Lambda$. Suppose v is a primitive null vector of $L \otimes \mathbb{R}$ of height 1, lying over $\ell \in \Lambda \otimes \mathbb{R}$, and that $D^2 = (\ell - \lambda h^{-1})^2$ satisfies $D^2 \leq R^2$, where R^2 is given by the table. Then there is another root r' of L , of the same height and length as r and also lying over λh^{-1} , such that either some reflection in r' preserves L and reduces the height of v , or else $D^2 = R^2$ and $\langle r'|v \rangle$ takes one of the values given in the table. Note that $R^2 = 1/|h|^2$ in all cases except that of long roots of height 1 in Gaussian lattices.

The ring \mathcal{R}	root length	height h	R^2	$\langle r' v \rangle$
\mathcal{G}	long	1	$\sqrt{3}$	$1 - \frac{\sqrt{3}}{2} + \frac{i}{2}$
		1	1	0
	short	$1+i$	$1/2$	0 or $h^{-1}i$
		2	$1/4$	0 or $h^{-1}i$
\mathcal{E}	long	1	1	$-\bar{\omega}$
		θ	$1/3$	$-h^{-1}\bar{\omega}$
	short	1	1	0
		θ	$1/3$	0
		2	$1/4$	0 or $h^{-1}\theta$
		2θ	$1/12$	0 or $h^{-1}\theta$
\mathcal{H}	long	1	1	$-\bar{\omega}$
	short	1	1	0
		$1+i$	$1/2$	0 or $\frac{1}{2}(1+i)$
		2	$1/4$	$\frac{1}{2}(1+ai+bj+ck)$ for $a, b, c \in \{0, 1\}$

Table 5.1. Summary of Lemmas 5.2 and 5.3, and generalizations thereof.

6. The reflection groups

This section is the heart of the paper: we will apply the results of Section 5 to find Lorentzian lattices that are reflective. We begin by providing a general criterion for a lattice to be reflective, and give a number of examples (Thms. 6.1–6.3). Then we will study in much greater detail the lattices $\mathcal{E}^{n,1}$ and $\mathcal{H}^{n,1}$ for small n (Lemma 6.4–Thm. 6.8), and also two high-dimensional examples, acting on $\mathbb{C}H^7$ and $\mathbb{H}H^5$ (Lemma 6.9–Thm. 6.14). At the end of the section we return to low dimensions, discussing the lattices $I_{1,1}^{\mathcal{R}}$ and $II_{1,1}^{\mathcal{G}}$. We begin with the most basic of our results:

Theorem 6.1. *Suppose Λ is a positive-definite \mathcal{R} -lattice which is spanned up to finite index by its roots and has covering radius ≤ 1 . Then $L = \Lambda \oplus II_{1,1}$ is reflective. Furthermore, if the covering radius is < 1 then any two primitive isotropic vectors of L are equivalent (up to a scalar) under $\text{Reflec } L$.*

Proof: According to Lemma 5.1, the stabilizer of ρ in $\text{Reflec } L$ has finite index in the stabilizer in $\text{Aut } L$. Now we study the $\text{Reflec } L$ orbits of primitive null vectors in L . Suppose v is such a vector, that it is not a multiple of ρ , and that it has minimal height in its $\text{Reflec } L$ orbit. Let ℓ be the element of $\Lambda \otimes \mathbb{Q}$ over which it lies, let λ be an element of Λ nearest ℓ , and let r be a short root of L of height 1 lying over λ (or a long root if $\mathcal{R} = \mathcal{G}$ and λ^2 is even). We must have $(\ell - \lambda)^2 \geq 1$, for else Lemma 5.3 (or Lemma 5.2 if $\mathcal{R} = \mathcal{G}$ and r is long) assures us that v is not of minimal height in its $\text{Reflec } L$ orbit. In particular, if Λ has covering radius < 1 then v cannot exist and we have proven that every null vector of L is equivalent under $\text{Reflec } L$ to a multiple of ρ . This is the second part of the theorem.

In case the covering radius of Λ is exactly 1, we can still deduce that there are only finitely many $\text{Reflec } L$ orbits of primitive null vectors in L . For if one cannot reduce the height of v by a reflection, then ℓ is a deep hole of Λ , and if λ is any vertex of the hole then there is a short root r of L of height 1 that lies over λ and is orthogonal to v . Now, v is determined up to a unit scalar by the point ℓ of $\Lambda \otimes \mathbb{Q}$ and the root r (lying over λ) to which it is orthogonal. Since the stabilizer of ρ in $\text{Reflec } L$ contains a finite-index subgroup of the translations of L , we may take r to lie in some fixed finite set of roots. Then ℓ is a deep hole nearest λ , for which there are only finitely many possibilities. That is, there are only finitely many $\text{Reflec } L$ orbits of primitive null vectors in L . The fact that $\text{Reflec } L$ has finite index in $\text{Aut } L$ follows from this, together with the fact that for one particular primitive null vector, namely ρ , its stabilizer in $\text{Reflec } L$ has finite index in its stabilizer in $\text{Aut } L$. \square

Essentially the same argument, using Lemma 5.2 in place of Lemma 5.3, proves the following theorem.

Theorem 6.2. *Suppose Λ is an even positive-definite \mathcal{G} -lattice which is spanned up to finite index by its roots and has covering radius $< \sqrt[4]{3}$. Then $L = \Lambda \oplus II_{1,1}$ is reflective and any two primitive null vectors of L are equivalent (up to a scalar) under $\text{Reflec } L$.* \square

Corollary 6.3. *Let Λ be any of the \mathcal{R} -lattices*

$\mathcal{G}, 2^{1/2}\mathcal{G}, D_4^{\mathcal{G}}, D_6^{\mathcal{G}} \text{ or } E_8^{\mathcal{G}}$	$\mathcal{E}^3 \text{ or } D_4(\theta)$	<i>if $\mathcal{R} = \mathcal{G}$,</i>
$\mathcal{E}, \mathcal{E}^2,$	$2^{1/2}\mathcal{H}, \mathcal{H}^2 \text{ or } E_8^{\mathcal{H}}$	<i>if $\mathcal{R} = \mathcal{E}$, or</i>
$\mathcal{H},$		<i>if $\mathcal{R} = \mathcal{H}$.</i>

Then L is reflective. Furthermore, if Λ appears in the first column of the list then any two primitive null vectors of L are equivalent (up to a scalar) under $\text{Reflec } L$.

Remark: The lattices appearing here are all described in Section 3. Thms. 6.6 and 6.8 give much more precise information about $\text{Reflec } L$ for $\Lambda = \mathcal{E}, \mathcal{E}^2, \mathcal{E}^3, \mathcal{H}$ or \mathcal{H}^2 .

Proof: All these lattices are spanned by their roots. The covering radii of the Gaussian lattices are $1/\sqrt{2}, 1, 1, \sqrt{3}/2$ and 1, and all but the first are even. The covering radii of the Eisenstein lattices are $1/\sqrt{3}, \sqrt{2}/3, 1$ and 1, and those of the Hurwitz lattices are $1/\sqrt{2}, 1, 1$ and 1. The result follows from Thms. 6.1 and 6.2. \square

We will now study in more detail the reflection groups of some low-dimensional selfdual Lorentzian lattices over \mathcal{E} and \mathcal{H} . If L is any lattice we will write $\text{Reflec}_0 L$ for the subgroup of $\text{Reflec } L$ generated by the reflections in the short roots of L .

Lemma 6.4. Suppose $\mathcal{R} = \mathcal{E}$, $\Lambda = \mathcal{E}^n$ ($n > 0$), and $L = \Lambda \oplus \Pi_{1,1}$. Then

- (i) $\text{Reflec}_0 L$ contains all the translations of L .
- (ii) $\text{Reflec}_0 L$ contains a transformation acting trivially on Λ and as ω on $\Pi_{1,1}$.
- (iii) The stabilizers of $\langle \rho \rangle$ in G and $\text{Aut } L$ coincide, where G is the group generated by $\text{Reflec}_0 L$ and the central involution $-I$ of L . Furthermore, $G \subseteq \text{Reflec } L$.

Proof: (i) By Lemma 5.1(i), $\text{Reflec } L$ contains a translation $T_{x,z}$ for each $x \in \mathcal{E}^n$. The proof shows that these translations actually lie in $\text{Reflec}_0 L$. Taking commutators as in Lemma 5.1(iii) shows that $\text{Reflec}_0 L$ contains all the reflections of L .

(ii) We have $T_{0,-\theta} \in \text{Reflec}_0 L$ by (i). Let F be the transformation composed of $T_{0,-\theta}$ followed by $(-\omega)$ -reflection in the short root $(0; 1, -\omega)$. It is obvious that F acts trivially on Λ and computation reveals that it acts on $\Pi_{1,1}^{\mathcal{R}}$ by left multiplication by the matrix

$$\begin{pmatrix} 0 & \bar{\omega} \\ \bar{\omega} & 0 \end{pmatrix}.$$

The square of this matrix is the scalar ω of $\Pi_{1,1}$, which proves the claim.

(iii) Since $\text{Reflec}_0 L$ contains the central involution of Λ , G contains the central involution J of $\Pi_{1,1}$. The biflection B in $b = (0, \dots, 0; 1, 1)$ acts trivially on Λ and on $\Pi_{1,1}$ as $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. One can check that $J = F^3 B$, where $F \in \text{Reflec}_0 L$ is as in (ii). This proves that $B \in G$ and also that $G = \langle \text{Reflec}_0 L, B \rangle$, hence $G \subseteq \text{Reflec } L$. We also note that since G contains J and also F^2 , it contains all the scalars of $\Pi_{1,1}$, so it suffices to show that G contains the full stabilizer in $\text{Aut } L$ of ρ . In light of (i) it suffices to merely show that G contains $\text{Aut } \Lambda$.

If $n = 1$ then $\text{Aut } \Lambda$ is generated by reflections in its short roots, as desired. If $n > 1$ then it suffices to prove that G contains the coordinate permutations with respect to the chosen basis of Λ . That is, we must show that G contains the biflections in vectors like $x = (1, -1, 0, \dots, 0; 0, 0)$. It suffices to show that x and b are equivalent under G . To see this, observe that $T_{(1,0,\dots,0),\theta/2}$ followed by F , followed by the scalar $-\omega$, followed by $T_{(-1,1,0,\dots,0),0}$, carries x to b . \square

Remarks: The condition $n > 0$ is necessary; one can show that $\text{Reflec } \Pi_{1,1}^{\mathcal{E}}$ contains no scalars except the identity.

Lemma 6.5. If r and r' are short roots in a lattice over $\mathcal{R} = \mathcal{E}$ or \mathcal{H} and $|\langle r | r' \rangle| = 1$, then r and r' are equivalent under the group generated by the reflections in them.

Proof: One checks that the $(-\omega)$ -reflections R and R' in r and r' satisfy the braid relation $RR'R = R'RR'$. (Because the Hermitian form is degenerate on the span of r and r' , one must check that this relation holds by using Eq. (2.1), not by just multiplying matrices for the actions of R and R' on the span of r and r' .) Rewriting this as $R'^{-1}RR' = RR'R^{-1}$ we see that R and R' are conjugate in the group they generate, which implies the lemma. \square

Remark: The proof suggests connections between the braid groups and complex reflection groups. This connection was first observed by Coxeter [16], and the braid groups play a central role in the work of Deligne and Mostow [18], Mostow [22] and Thurston [24]. They are also important in work of the author, J. Carlson and D. Toledo [2], [3] on moduli of cubic surfaces.

Theorem 6.6. Let $\mathcal{R} = \mathcal{E}$, $\Lambda = \mathcal{E}^n$ and $L = \Lambda \oplus \Pi_{1,1}^{\mathcal{E}}$.

- (i) If $n = 1$ then $\text{Reflec}_0 L$ acts with exactly 2 orbits of primitive null vectors, represented by $\pm \rho$. If $n = 2$ or 3 then $\text{Reflec}_0 L$ acts transitively on the primitive null vectors of L .
- (ii) If $n = 1, 2$ or 3 then $\text{Aut } L = \text{Reflec } L = \text{Reflec}_0 L \times \{\pm I\}$.

Proof: First we show that $\text{Reflec}_0 L$ acts transitively on the 1-dimensional primitive null lattices in L . For $n = 1$ or 2 this follows from Thm. 6.3. So suppose $n = 3$ and that $v \in L$ is a primitive null vector not proportional to ρ and of smallest height in its orbit under $\text{Reflec}_0 L$. Since the covering radius of \mathcal{E}^3 is 1, Lemma 5.3(ii) implies that v is orthogonal to a short root of height 1. By applying a translation (Lemma 6.4(i)) we may suppose that this root is $r_1 = (0, 0, 0; 1, -\omega)$. Taking $r_2 = (0, 0, 1; 0, 1)$ and $r_3 =$

$(0, 0, 1; 0, 0)$ we see that $\langle r_1 | r_2 \rangle = \langle r_2 | r_3 \rangle = 1$, so Lemma 6.5 shows that r_1 is equivalent to r_3 under $\text{Reflec}_0 L$. Thus v is equivalent to an element of r_3^\perp , which is a copy of $\mathcal{E}^2 \oplus \Pi_{1,1}^\mathcal{E}$. Applying the $n = 2$ case, we see that $\text{Reflec}_0 L$ acts transitively on the primitive null sublattices of L .

It follows from Lemma 6.4(iii) that $\text{Aut } L$ is generated by $\text{Reflec}_0 L$ and $\{\pm I\}$. Now we will show that $-I \notin \text{Reflec}_0 L$, which will establish the equality $\text{Aut } L = \text{Reflec}_0 L \times \{\pm I\}$. Since $-I \in \text{Reflec } L$ by Lemma 6.4(iii) we will have proven (ii). To prove $-I \notin \text{Reflec}_0 L$ we must consider the finite vector space $V = L/L\theta$ over $\mathbb{F}_3 = \mathcal{E}/\theta\mathcal{E}$; we write q for both natural maps $L \rightarrow V$ and $\mathcal{E} \rightarrow \mathbb{F}_3$. The Hermitian form on L gives rise to a symmetric bilinear form on V , given by $\langle q(v) | q(w) \rangle = q(\langle v | w \rangle)$. Since $\Pi_{1,1}^\mathcal{E} \cong I_{1,1}^\mathcal{E}$, L and hence V admits an orthogonal basis with $n + 1$ vectors of norm 1 and one of norm -1 . There is a homomorphism called the spinor norm from $\text{Aut } V$ to the set $\{\pm 1\}$ of nonzero square classes in \mathbb{F}_3 . This is characterized by the property that the reflection in a vector of V of norm ± 1 has spinor norm ± 1 . It is clear that $-I$ acts on V with spinor norm -1 . Since a reflection of L in a short root r acts on V either trivially or as the reflection in $q(r)$, of norm $q(r^2) = 1$, every element of $\text{Reflec}_0 L$ acts on V with spinor norm $+1$. Hence $-I \notin \text{Reflec}_0 L$, as desired. This also characterizes $\text{Reflec}_0 L$ as the subgroup of $\text{Aut } L$ whose elements act on V with spinor norm $+1$.

Now we will establish (i). By the first part of the proof it suffices to determine which multiples of ρ are equivalent to each other. By Lemma 6.4(ii) it suffices to determine whether $\pm\rho$ are equivalent. If $n = 1$ then they are inequivalent, because the stabilizers of ρ in $\text{Reflec}_0 L$ and $\text{Aut } L$ coincide. If ρ and $-\rho$ were equivalent then we would have transitivity on primitive null vectors and $\text{Reflec}_0 L = \text{Aut } L$ would follow. Since this is not true, ρ and $-\rho$ are inequivalent. On the other hand, if $n = 2$ or 3 , then $\pm\rho$ are equivalent. To see this, apply the product of $-I$ and the biflection in any long root of L that is orthogonal to ρ . The product exchanges $\pm\rho$, and by spinor norm considerations it lies in $\text{Reflec}_0 L$. \square

Remark: The case $n = 3$ arises in algebraic geometry: the quotient of $\mathbb{C}H^4$ by $\text{Reflec}_0 I_{4,1}^\mathcal{E}$ may be identified with the moduli space of stable cubic surfaces in $\mathbb{C}P^3$. One can also construct the moduli space of marked stable cubic surfaces by taking the quotient of $\mathbb{C}H^4$ by the congruence subgroup of $\text{Reflec}_0 I_{4,1}^\mathcal{E}$ associated to the prime $\theta \in \mathcal{E}$. The quotient of $\text{Reflec}_0 I_{4,1}^\mathcal{E}$ by this normal subgroup is the E_6 Weyl group, also known as “the group of the 27 lines on a cubic surface”. See [2] for details.

Lemma 6.7. *Suppose $\mathcal{R} = \mathcal{H}$, $\Lambda = \mathcal{H}^n$ ($n > 0$) and $L = \Lambda \oplus \Pi_{1,1}$. Then*

- (i) *$\text{Reflec}_0 L$ contains a translation $T_{x,z}$ for each $x \in \Lambda$, and also the central translations $T_{0,ai+bj+ck}$ with $a \equiv b \equiv c \pmod{2}$. In particular, coset representatives for the translations of $\text{Reflec}_0 L$ in those of $\text{Aut } L$ may be taken from $\{T_{0,0}, T_{0,i}, T_{0,j}, T_{0,k}\}$.*
- (ii) *$\text{Reflec}_0 L$ contains transformations acting trivially on Λ and on $\Pi_{1,1}$ by left scalar multiplication by any given unit of \mathcal{H} .*
- (iii) *The stabilizer of $\langle \rho \rangle$ in $\text{Reflec}_0 L$ has index ≤ 4 in the stabilizer in $\text{Aut } L$; coset representatives may be taken from the set given in (i).*

Proof: (i) The first part follows immediately from Lemma 5.1(i). The second part may be obtained by taking commutators: if $\lambda, \lambda' \in \Lambda$ and $z, z' \in \text{Im } \mathbb{K}$ are such that $T_{\lambda,z}, T_{\lambda',z'} \in \text{Reflec}_0 L$, then $T_{0,\pm 2\text{Im}(\lambda|\lambda')} \in \text{Reflec}_0 L$ by Eq. (4.5). Since Λ contains vectors λ and λ' with $\langle \lambda | \lambda' \rangle = \alpha$ for any given unit α of \mathcal{H} , we see that $\text{Reflec}_0 L$ contains $T_{0,2i}, T_{0,2j}, T_{0,2k}$ and $T_{0,i+j+k}$. These generate the group of central translations given in the statement of the lemma.

(ii) The argument of Lemma 6.4(ii) shows that $\text{Reflec}_0 L$ contains an element acting trivially on Λ and on $\Pi_{1,1}$ as left-multiplication by ω . Taking conjugates of this by the group $\text{Aut } \mathcal{H}$ acting on $\Pi_{1,1}$, which normalizes $\text{Reflec}_0 L$ even though it doesn't act \mathcal{H} -linearly, we see that $\text{Reflec}_0 L$ contains elements acting on $\Pi_{1,1}$ as left-multiplication by any of the units $(-1 \pm i \pm j \pm k)/2$ of \mathcal{H} . These generate the group of all units of \mathcal{H} , proving (ii).

(iii) Follows immediately from (i) and (ii) and the arguments given for Lemma 6.4(iii). Note the curious fact that $\text{Reflec}_0 L$ contains the biflection B , which it did not in the Eisenstein case. \square

Theorem 6.8. *Let $\mathcal{R} = \mathcal{H}$, $\Lambda = \mathcal{H}^n$ for $n = 1$ or 2 , and $L = \Lambda \oplus \Pi_{1,1}^\mathcal{H}$. Then $\text{Reflec}_0 L$ acts transitively on the primitive null vectors of L and has index at most 4 in $\text{Aut } L$; coset representative may be taken from $\{T_{0,0}, T_{0,i}, T_{0,j}, T_{0,k}\}$.*

Proof: We first claim that $\text{Reflec}_0 L$ acts transitively on primitive null lattices in L . For $n = 1$ this

follows from Thm. 6.3. For $n = 2$ it follows from an argument similar to the $n = 3$ case of Thm. 6.6. That is, the covering radius of $\Lambda = \mathcal{H}^2$ is 1, so if $v \in L$ is primitive, isotropic and of smallest height in its orbit under $\text{Reflec}_0 L$ then by Lemma 5.3(ii) we see that v is either proportional to ρ or orthogonal to a short root r_1 of height 1. In the latter case, after applying a translation of $\text{Reflec}_0 L$, courtesy of Thm. 6.7(i), we may take $r = (0, 0; 1, x - \omega)$, where x is one of 0, i, j and k. In any of these cases, upon taking $r_2 = (0, 1; 0, 1)$ and $r_3 = (0, 1; 0, 0)$ we have $\langle r_1 | r_2 \rangle = \langle r_2 | r_3 \rangle = 1$. By Lemma 6.5, v is equivalent under $\text{Reflec}_0 L$ to an element of r_3^\perp . Since r_3^\perp is a copy of $\mathcal{H}^1 \oplus \Pi_{1,1}^{\mathcal{H}}$, the transitivity follows from the case $n = 1$.

The transitivity on primitive null vectors follows from Thm. 6.7(ii). The rest of the theorem follows from Lemma 6.7(iii). \square

Now we move on to higher-dimensional examples—we will construct a group acting on $\mathbb{C}H^7$ and another acting on $\mathbb{H}H^5$. These arise from our basic construction by taking $\Lambda = \Lambda_6^\mathcal{E}$ or $\Lambda_4^{\mathcal{H}}$.

Lemma 6.9. *Suppose $v, r_1, \dots, r_m \in \mathbb{K}^n \oplus \mathbb{K}^{1,1}$ lie over $\ell, \lambda_1, \dots, \lambda_m \in \mathbb{K}^n$ respectively. Suppose $v^2 = 0$, that $\langle r_i | v \rangle = 0$ for all i , and that the vectors $\lambda_i - \ell$ are linearly independent in \mathbb{K}^n . Then the images of the r_i in $v^\perp / \langle v \rangle$ are linearly independent.*

Proof: We may obviously replace v and the r_i by any scalar multiples of themselves and so suppose that they have height 1. Thus $v = (\ell; 1, ?)$ and $r_i = (\lambda_i; 1, ?)$ where the question marks denote irrelevant (and possibly distinct) elements of \mathbb{K} . Let T be the translation carrying v to $(0; 1, 0)$, so $T(r_i) = (\lambda_i - \ell; 1, 0)$. (The last coordinate vanishes because $\langle T(r_i) | Tv \rangle = 0$.) Since the image of $T(r_i)$ in $(Tv)^\perp / \langle Tv \rangle$ may be identified with its first coordinate, namely $\lambda_i - \ell$, the images of the $T(r_i)$ in $(Tv)^\perp / \langle Tv \rangle$ are linearly independent. The lemma follows immediately. \square

Lemma 6.10. $\Lambda_6^\mathcal{E} \otimes \mathbb{R}$ is covered by the closed balls of radius 1 centered at points of $\Lambda_6^\mathcal{E}$, together with those of radius $1/\sqrt{3}$ centered at points $\lambda\theta^{-1}$ with $\lambda \in \Lambda_6^\mathcal{E}$ and $\lambda^2 \equiv 1 \pmod{3}$.

Proof: Section 7 of [14] defines a linear “gluing map” $g : \frac{1}{\theta}\Lambda_6^\mathcal{E}/\Lambda_6^\mathcal{E} \rightarrow \frac{1}{\theta}\Lambda_6^\mathcal{E}/\Lambda_6^\mathcal{E}$ with the property that the Leech lattice Λ_{24} , scaled down by $2^{1/2}$, is the real form of the lattice of vectors $(x_1, x_2) \in (\frac{1}{\theta}\Lambda_6^\mathcal{E})^2$ satisfying $g(x_1 + \Lambda_6^\mathcal{E}) = x_2 + \Lambda_6^\mathcal{E}$. Identifying $\Lambda_6^\mathcal{E}$ with the set of such (x_1, x_2) with $x_2 = 0$, we see that the only points of $2^{-1/2}\Lambda_{24}$ at distance < 1 from $\Lambda_6^\mathcal{E} \otimes \mathbb{R}$ are those in $\Lambda_6^\mathcal{E}$ and those of the form $(x_1\theta^{-1}, x_2\theta^{-1})$ with x_2 a minimal vector of $\Lambda_6^\mathcal{E}$ (a long root). The definition of g (see [14]) shows that $x_1^2 \equiv 1 \pmod{3}$ if and only if there is a long root x_2 of $\Lambda_6^\mathcal{E}$ such that $(x_1\theta^{-1}, x_2\theta^{-1}) \in 2^{-1/2}\Lambda_{24}$. By [12], the covering radius of $2^{-1/2}\Lambda_{24}$ is 1. Therefore the balls of radius 1 centered at the points of $\Lambda_6^\mathcal{E}$ and at the points $(x_1\theta^{-1}, x_2\theta^{-1})$ with $x_1^2 \equiv 1 \pmod{3}$ and $x_2^2 = 2$ cover $\Lambda_6^\mathcal{E} \otimes \mathbb{R}$. Computing the radius of the intersection of a ball of the second family with $\Lambda_6^\mathcal{E} \otimes \mathbb{R}$ yields the lemma. \square

Theorem 6.11. Let $\Lambda = \Lambda_6^\mathcal{E}$ and $L = \Lambda \oplus \Pi_{1,1}^\mathcal{E} \cong I_{7,1}^\mathcal{E}$.

- (i) If $v \in L$ is primitive, isotropic and not equivalent to a multiple of ρ under $\text{Reflec}_0 L$, then $v^\perp / \langle v \rangle \cong \mathcal{E}^6$.
- (ii) $\text{Aut } L = \text{Reflec } L$. In particular, L is reflective.

Proof: (i) Suppose that v is a primitive isotropic vector of smallest height in its orbit under $\text{Reflec}_0 L$. Suppose v is not a multiple of ρ , so that it lies over some $\ell \in \Lambda \otimes \mathbb{R}$. By Lemma 5.3 and the minimality of the height of v , ℓ lies at distance ≥ 1 from each lattice point $\lambda \in \Lambda$ and at distance $\geq 3^{-1/2}$ from each $\lambda\theta^{-1}$ with $\lambda \in \Lambda$ and $\lambda^2 \equiv 1 \pmod{3}$. By Lemma 6.10 the set S of such points in $\Lambda \otimes \mathbb{R}$ is discrete. Let μ_1, \dots, μ_n be the elements of Λ with $(\ell - \mu_i)^2 = 1$ and let ν_1, \dots, ν_m be those vectors of the form $\lambda\theta^{-1}$ with $\lambda \in \Lambda$ and $\lambda^2 \equiv 1 \pmod{3}$ such that $(\ell - \nu_i)^2 = 1/3$. Over each μ_i (resp. ν_i) there is a short root of L , say r_i (resp. s_i), of height 1 (resp. θ). By Lemma 5.3, we may suppose that the r_i and s_i are orthogonal to v . Because S is discrete the vectors $\mu_i - \ell$ and $\nu_i - \ell$ span $\Lambda \otimes \mathbb{R}$, and therefore there are 6 among them that are linearly independent over \mathbb{C} . By Lemma 6.9 this implies that among the images in $v^\perp / \langle v \rangle$ of the vectors r_i and s_i are 6 short roots that are linearly independent over \mathcal{E} . Since $v^\perp / \langle v \rangle$ is positive-definite, it follows that $v^\perp / \langle v \rangle \cong \mathcal{E}^6$.

(ii) We have seen the transitivity of $\text{Reflec}_0 L$ on the primitive null sublattices that, like $\langle \rho \rangle$, are orthogonal to no short roots. Since $L \cong I_{7,1}^\mathcal{E} \cong \mathcal{E}^6 \oplus \Pi_{1,1}$, Lemma 6.4(iii) implies that $\langle \text{Reflec}_0 L, -I \rangle$ lies in $\text{Reflec } L$ and contains the scalars of L . Therefore $\text{Reflec } L$ acts transitively on the primitive null vectors that, like ρ , are orthogonal to no short roots. Now it suffices to show that $\text{Reflec } L$ contains the full stabilizer of

ρ . Since Λ is selfdual and spanned by its long roots, Lemma 5.1(ii) and (iii) imply that $\text{Reflec } L$ contains all the translations of L . Since $\text{Aut } \Lambda$ is a reflection group, the proof is complete. \square

Remark: it would be nice to understand the groups $\text{Reflec}_0 I_{7,1}^{\mathcal{E}}$ and $\text{Reflec}_0 I_{5,1}^{\mathcal{E}}$.

We now study the quaternionic lattice $I_{5,1}^{\mathcal{H}}$. The analysis is surprisingly similar to our study of $I_{7,1}^{\mathcal{E}}$. In particular, Lemma 6.12 below is very similar to Lemma 6.10. Section 5 of [13] describes an embedding of the real form BW_{16} of $2^{1/2}\Lambda_4^{\mathcal{H}}$ into the Leech lattice Λ_{24} . When we refer to concepts involving Λ_{24} while discussing $\Lambda_4^{\mathcal{H}}$, we implicitly refer to this embedding. (Up to isometry of Λ_{24} , there is only one embedding.)

Lemma 6.12. $\Lambda_4^{\mathcal{H}} \otimes \mathbb{R}$ is covered by the closed balls of radius 1 centered at points of $\Lambda_4^{\mathcal{H}}$, together with those of radius $1/\sqrt{2}$ centered at points $\lambda(1+i)^{-1}$ with $\lambda \in \Lambda_4^{\mathcal{H}}$ of odd norm. Any point of $\Lambda_4^{\mathcal{H}} \otimes \mathbb{R}$ not in the interior of one of these balls is a deep hole of $2^{-1/2}\Lambda_{24}$.

Proof: The orthogonal complement of $\Lambda_4^{\mathcal{H}}$ in $2^{-1/2}\Lambda_{24}$ is a copy of the E_8 lattice. Properties of the embedding are described in [13] and include the following. If $(x, y) \in (\Lambda_4^{\mathcal{H}} \otimes \mathbb{R}) \times (E_8 \otimes \mathbb{R})$ lies in $2^{-1/2}\Lambda_{24}$, then $y \in \frac{1}{2}E_8$ and hence has norm $n/2$ for some nonnegative integer n . We write $B(x, y)$ for the ball of radius 1 with center $(x, y) \in 2^{-1/2}\Lambda_{24}$. Only if the norm of y is 0 or $1/2$ does the interior of $B(x, y)$ meet $\Lambda_4^{\mathcal{H}} \otimes \mathbb{R}$. Those x for which $(x, y) \in 2^{-1/2}\Lambda_{24}$ for some y of norm $1/2$ are exactly the deep holes of $\Lambda_4^{\mathcal{H}}$. By Thm. 3.1, the set of such x coincides exactly with $\{\lambda(1+i)^{-1} \mid \lambda \in \Lambda_4^{\mathcal{H}}, \lambda^2 \equiv 1 \pmod{2}\}$. For such (x, y) , the ball $B(x, y)$ meets $\Lambda_4^{\mathcal{H}} \otimes \mathbb{R}$ in a ball of radius $2^{-1/2}$. The theorem follows from the fact that the covering radius of $2^{-1/2}\Lambda_{24}$ is 1 (see [12]). \square

Lemma 6.13. Any deep hole of Λ_{24} that lies in $BW_{16} \otimes \mathbb{R}$ has a vertex in BW_{16} .

Proof: The natural language for discussing the deep holes of Λ_{24} is that of affine Coxeter-Dynkin diagrams, using the slightly nonstandard conventions of [12]. If h is a deep hole of Λ_{24} then its vertices v_i lie at distance $\sqrt{2}$ from h , and we define the diagram Δ of h to be the graph whose vertices are the v_i , with v_i and v_j unjoined, singly joined or doubly joined according to whether $(v_i - v_j)^2$ is 4, 6 or 8. Each component of Δ is an affine diagram of type A_n , D_n or E_n . For the rest of the proof we will take h as the origin. The definition of Δ and the fact that $(v_i - h)^2 = 2$ for all i means that the inner product of v_i and v_j is 0, -1 or -2 according to whether the corresponding vertices of Δ are unjoined, singly joined, or doubly joined. It follows that the subspaces spanned by different components of Δ are orthogonal, and that the vertices corresponding to each component form a system of simple roots of the corresponding type, together with the lowest root, which corresponds to the extending node in the diagram. Now, BW_{16} is the fixed-point set of an involution ϕ of Λ_{24} . Since ϕ fixes h , it acts on Δ . We will show that ϕ preserves a vertex v of Δ , which forces v to lie in BW_{16} , which proves the lemma.

For each component C of Δ we write S_C for the corresponding subspace of \mathbb{R}^{24} . If ϕ preserves C then we write F_C for the subspace of S_C fixed pointwise by ϕ . We write F for the subspace of \mathbb{R}^{24} fixed pointwise by ϕ . It is easy to see that

$$\dim F = \sum_{\phi(C)=C} \dim F_C + \sum_{\phi(C) \neq C} \frac{\dim S_C}{2}, \quad (6.1)$$

where the sums are over the components C of Δ that are (resp. are not) preserved by ϕ . If ϕ preserves C then its action on C determines F_C . The explicit description in terms of root systems allows one to deduce that $\dim F_C$ equals the number of vertices of C preserved by ϕ , plus half the number not preserved, minus 1. It follows that if ϕ permutes the v_i freely then each term in each sum in Eq. (6.1) is bounded by $\frac{1}{2} \dim S_C$. Since $\sum_C \dim S_C = 24$ we would obtain $\dim F \leq 12$, which is impossible since $BW_{16} \subseteq F$. \square

Remark: A more involved analysis shows that any deep hole of Λ_{24} lying in $BW_{16} \otimes \mathbb{R}$ has at least nine vertices in BW_{16} , and that this bound cannot be improved.

Theorem 6.14. Let $\Lambda = \Lambda_4^{\mathcal{H}}$ and $L = \Lambda \oplus II_{1,1}^{\mathcal{H}} \cong I_{5,1}^{\mathcal{H}}$.

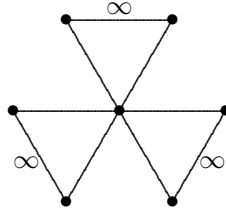
- (i) If $v \in L$ is a primitive isotropic vector not equivalent under $\text{Reflec}_0 L$ to a multiple of ρ then $v^\perp / \langle v \rangle$ contains a short root.
- (ii) The index of $\text{Reflec } L$ in $\text{Aut } L$ is at most 4, so that L is reflective. More precisely, coset representatives for $\text{Reflec } L$ in $\text{Aut } L$ may be taken from $\{T_{0,0}, T_{0,i}, T_{0,j}, T_{0,k}\}$.

Proof: (i) This is very similar to the proof of Thm. 6.11(i). Suppose that v is a primitive isotropic vector of L of smallest height in its orbit under $\text{Reflec } L$. Suppose v is not a multiple of ρ , so that v lies over some $\ell \in \Lambda \otimes \mathbb{R}$. By Lemma 5.3, ℓ lies at distance ≥ 1 from each lattice point $\lambda \in \Lambda$ and at distance $\geq 2^{-1/2}$ from each point $\lambda(1+i)^{-1}$ with $\lambda \in \Lambda$ and $\lambda^2 \equiv 1 \pmod{2}$. By Lemma 6.12, ℓ must be a deep hole of $2^{-1/2}\Lambda_{24}$. By Lemma 6.13 there is a vertex $\lambda \in \Lambda_4^{\mathcal{H}}$ of the hole with $(\ell - \lambda)^2 = 1$. There is a short root of L lying over λ , and by Lemma 5.3 there is also one orthogonal to v .

(ii) Since Λ is selfdual and spanned by its long roots, Lemma 5.1(ii) shows that $\text{Reflec } L$ contains a translation $T_{x,z}$ for each $x \in \Lambda$. Taking commutators as in Lemma 6.7(i) shows that $\text{Reflec } L$ contains the central translations $T_{0,ai+bj+ck}$ with $a \equiv b \equiv c \pmod{2}$. Then the proof of Lemma 6.7(ii) shows that $\text{Reflec } L$ contains elements acting on $\mathcal{H}_{1,1}$ by left-multiplication by the units of \mathcal{H} . Together with (i) this proves the transitivity of $\text{Reflec } L$ on primitive null vectors that, like ρ , are orthogonal to no short roots. Then (ii) follows from the facts that $\text{Aut } \Lambda$ is a reflection group and $\text{Reflec } L$ contains the translations just discussed. \square

We close this section by returning to low dimensions, studying the 2-dimensional selfdual Lorentzian lattices. If $\mathcal{R} = \mathcal{E}$ or \mathcal{G} then one can obtain very explicit descriptions of the groups by drawing pictures in $\mathbb{C}H^1 \subseteq \mathbb{C}P^1$. In particular, if we represent a point $(a,b) \in \mathcal{H}_{1,1}$ by $a/b \in \mathbb{C}P^1$ then the hyperbolic space becomes the right half-plane and ρ the point at infinity. It is easy to find the points of $\mathbb{C}P^1$ corresponding to the roots of L of small height, and then one can work out the group $\text{Reflec } L$. For example, one can check that $\text{Reflec } \mathcal{H}_{1,1}^{\mathcal{E}}$ acts as the triangle group $(2,6,\infty)$. One can also show that $\text{Aut } \mathcal{H}_{1,1}^{\mathcal{G}}$ acts on $\mathbb{C}H^1$ as $(2,3,\infty)$ and its subgroup of index 2 consisting of elements with determinant $+1$ is conjugate in $\text{GL}_2(\mathcal{G})$ to $\text{SL}_2\mathbb{Z}$. The group $\text{Reflec } \mathcal{H}_{1,1}^{\mathcal{G}}$ is generated by 3 biflections, which act by rotations by π around the three finite corners of a quadrilateral in $\mathbb{C}H^1$ with corner angles $\pi/2, \pi/2, \pi/2$ and π/∞ . For completeness we mention that $\text{Aut } \mathcal{H}_{1,1}^{\mathcal{G}}$ acts on $\mathbb{C}H^1$ as $(2,4,\infty)$, and its reflection subgroup acts as $(4,4,\infty)$. See [17] for descriptions of the groups (p,q,r) and other information.

One can also treat the quaternionic case: an adaptation of the argument of [4, Thm. 5.3(i)] shows that $\text{Reflec } \mathcal{H}_{1,1}^{\mathcal{H}}$ acts on $\mathbb{H}H^1 \cong \mathbb{R}H^4$ as the rotation subgroup of the real hyperbolic reflection group with the Coxeter diagram below. Note that the 6 outer nodes generate an affine reflection group, so this graph is a special case of the usual procedure of “hyperbolizing” an affine reflection group by adjoining an extra node.



7. Enumeration of selfdual lattices

As we explain below, the orbits of primitive isotropic lattices in the Lorentzian lattice $I_{n+1,1}^{\mathcal{R}}$ are in natural 1-1 correspondence with the equivalence classes of positive-definite selfdual lattices of dimension n over \mathcal{R} . This means that one may classify such lattices by studying $\text{Aut } I_{n+1,1}^{\mathcal{R}}$. Since we have made such a study in the previous section, in terms of the geometry of various special lattices, we can now classify selfdual lattices in low dimensions. We begin with an analogue of a result well-known for lattices over \mathbb{Z} .

Theorem 7.1. *An indefinite selfdual lattice L over $\mathcal{R} = \mathcal{E}$ or \mathcal{H} is characterized up to isometry by its dimension and signature. An indefinite selfdual lattice L over $\mathcal{R} = \mathcal{G}$ is characterized up to isometry by its dimension, signature, and whether it is even; if L is even with signature (n,m) then $n-m$ is divisible by 4.*

Proof: First we show that any indefinite selfdual \mathcal{R} -lattice L contains an isotropic vector. If $\mathcal{R} = \mathcal{H}$, or if $\dim L > 2$, then the real form of $L \otimes \mathbb{Q}$ is an indefinite rational bilinear form of rank > 4 , so Meyer’s theorem [20, Chap. 2] asserts the existence of an isotropic vector. If $\dim L = 2$ and $\mathcal{R} = \mathcal{G}$ or \mathcal{E} , then we consider the 2×2 matrix of inner products of the elements of a basis for L . This may be diagonalized by row and column operations over $\mathcal{R} \otimes \mathbb{Q}$ to a diagonal matrix $[a, -a^{-1}]$ with $a \in \mathbb{Q}$. (Each term is real because the matrix is Hermitian, and each determines the other because the determinant is -1 .) Then the vector $(1,a)$ is isotropic. Having obtained an isotropic vector in $L \otimes \mathbb{Q}$, we may multiply by a scalar to obtain one in L .

If L is odd then the proof of Thm. 4.3 in [20, Chap. 2] applies, and $L \cong I_{n,m}^{\mathcal{R}}$ for some n and m . This completes the proof of the first claim, since any selfdual lattice over \mathcal{E} or \mathcal{H} is odd: if $v, w \in L$ satisfy $\langle v|w \rangle = \omega$ then v^2, w^2 and $(v+w)^2$ cannot all have the same parity. This also proves that an odd indefinite selfdual Gaussian lattice is characterized by its dimension and signature.

One may construct lattices N from an odd Gaussian lattice M by considering the sublattice M^e consisting of the elements of M of even norm, and considering the 3 lattices N such that $M_e' \subseteq N \subseteq M_e$. When M is $I_{1,1}^{\mathcal{G}}$, then N may be chosen to be $II_{1,1}^{\mathcal{G}}$. Now consider an indefinite even selfdual \mathcal{G} -lattice L . We know that L contains an isotropic vector, and as in [20] there is a decomposition $L = \Lambda \oplus II_{1,1}^{\mathcal{G}}$. We see that L arises by applying the construction above to the odd selfdual lattice $\Lambda \oplus I_{1,1}^{\mathcal{G}}$. Since $\Lambda \oplus I_{1,1}^{\mathcal{G}}$ is isomorphic to $I_{n,m}^{\mathcal{G}}$, it is clear that all possible L can be constructed by applying our construction to the various $I_{n,m}^{\mathcal{G}}$. No even lattices arise unless $n - m \equiv 0 \pmod{4}$, when two isometric ones do. \square

Special cases of Thm. 7.1 are $I_{7,1}^{\mathcal{E}} \cong \Lambda_6^{\mathcal{E}} \oplus II_{1,1}^{\mathcal{E}}$ and $I_{5,1}^{\mathcal{H}} \cong \Lambda_4^{\mathcal{H}} \oplus II_{1,1}^{\mathcal{H}}$, which are the lattices studied in Thms. 6.11 and 6.14. Thm. 7.1 also provides the correspondence mentioned above: if V is a primitive isotropic lattice in $I_{n+1,1}^{\mathcal{R}}$ then it is easy to check that V^\perp/V is an n -dimensional positive-definite selfdual lattice, and that this establishes a one-to-one correspondence between orbits of primitive isotropic lattices of $I_{n+1,1}^{\mathcal{R}}$ and isometry classes of selfdual positive-definite lattices in dimension n . Similarly, the orbits of primitive isotropic lattices of $II_{n+1,1}^{\mathcal{G}}$ correspond to the classes of positive-definite even selfdual Gaussian lattices of dimension n .

Theorem 7.2. *The positive-definite selfdual \mathcal{E} -lattices in dimensions ≤ 6 are \mathcal{E}^n and $\Lambda_6^{\mathcal{E}}$. The positive-definite selfdual \mathcal{H} -lattices in dimensions ≤ 4 are \mathcal{H}^n and $\Lambda_4^{\mathcal{H}}$. The positive-definite even selfdual \mathcal{G} -lattices in dimensions ≤ 4 are $\{0\}$ and $E_8^{\mathcal{G}}$.*

Proof: By Thm. 6.11(i), any primitive null vector v of $I_{7,1}^{\mathcal{E}}$ satisfies $v^\perp/\langle v \rangle \cong \Lambda_6^{\mathcal{E}}$ or $v^\perp/\langle v \rangle \cong \mathcal{E}^6$; the first claim follows immediately. To see the last claim, suppose that Λ is an even selfdual \mathcal{G} -lattice of dimension ≤ 4 . By the signature condition, the dimension is either 0 or 4. In the latter case the isomorphism $\Lambda \cong E_8^{\mathcal{G}}$ follows from the equivalence of any two primitive null vectors in $II_{5,1}^{\mathcal{G}} = E_8^{\mathcal{G}} \oplus II_{1,1}$ (Thm. 6.3).

We will only sketch the quaternionic case. By Thm. 6.14(i), any 4-dimensional positive-definite selfdual \mathcal{H} -lattice Λ besides $\Lambda_4^{\mathcal{H}}$ has a short root. We claim that in fact Λ has a pair of independent (hence orthogonal) short roots. This follows from the remark after Lemma 6.13: if ℓ is a deep hole of $2^{-1/2}\Lambda_{24}$ lying in $\Lambda_4^{\mathcal{H}} \otimes \mathbb{R}$ then it has 9 vertices v_i in $\Lambda_4^{\mathcal{H}}$. By considering the hole diagram of ℓ one can show that the $v_i - \ell$ span a space of real dimension ≥ 5 , hence of dimension ≥ 2 over \mathbb{H} . Then the argument of Thm. 6.14(i) establishes the claim. Therefore Λ is the direct sum of \mathcal{H}^2 and a two-dimension selfdual \mathcal{H} -lattice. The second summand must also be \mathcal{H}^2 , by the treatment of $I_{3,1}^{\mathcal{H}}$ in Thm. 6.8. \square

These results have been obtained before, by very different means. Feit [19] found examples of many positive-definite selfdual \mathcal{E} -lattices. He derived a version of the mass formula to verify that his list was complete for dimensions $n \leq 12$. Conway and Sloane [14, Thm. 3] provide a nice proof of this classification in dimensions $n \leq 6$ based on theta series and modular forms. (Their proof does not apply for $6 < n < 12$: in the second-to-last sentence of the proof, “12” should be replaced by “7”.) Although selfdual \mathcal{G} -lattices have not been tabulated, it would be easy (and boring) to enumerate them through dimension 12 by using the fact that the real form of a selfdual \mathcal{G} -lattice is selfdual over \mathbb{Z} . An enumeration of positive-definite selfdual \mathcal{H} -lattices for dimensions $n \leq 7$ has recently been completed by Bachoc [5] and for $n = 8$ by Bachoc and Nebe [6]. These enumerations are based on a generalization of Kneser’s notion of “neighboring” lattices, together with a suitable version of the mass formula.

8. Comparison with the groups of Deligne and Mostow

In this section we justify the word “new” in our title, by showing that our largest three reflection groups do not appear on the lists of Mostow [23] and Thurston [24]. Deligne and Mostow [18] and Mostow [22] constructed 94 reflection groups acting on $\mathbb{C}H^n$ for various $n = 2, \dots, 9$ by considering the monodromy of hypergeometric functions. Thurston [24] constructed the same set of groups in terms of moduli of flat metrics (with specified sorts of singularities) on the sphere S^2 . We will generally refer to these groups as the DM groups. We show here (Thm. 8.4) that none of the groups $\text{Reflec } I_{n,1}^{\mathcal{E}}$ ($n \geq 4$) or $\text{Reflec } II_{4n+1,1}^{\mathcal{G}}$ ($n \geq 1$) appear on their lists. In particular, our groups $\text{Reflec } I_{7,1}^{\mathcal{E}}$, $\text{Reflec } I_{4,1}^{\mathcal{E}}$ and $\text{Reflec } II_{5,1}^{\mathcal{G}}$ are new. We will also

identify $\text{Reflec } I_{3,1}^{\mathcal{E}}$ with one of the DM groups. We leave open the question of whether our other groups appear on their lists and also the question of commensurability.

We will distinguish our groups from the DM groups by considering the orders of the reflections in the groups. We begin by showing that the only reflections of the selfdual lattices are the obvious ones, a result well-known for lattices over \mathbb{Z} .

Lemma 8.1. *Any reflection R of a selfdual lattice M over $\mathcal{R} = \mathcal{E}$ or \mathcal{G} is either a reflection in a lattice vector of norm ± 1 or a biflection in a lattice vector of norm ± 2 .*

Proof: By considering the determinant of R we discover that its only nontrivial eigenvalue is a unit of \mathcal{R} , so M contains an element of the corresponding eigenspace, so R is the α -reflection in some lattice vector v , where α is a unit of \mathcal{R} . Taking v to be primitive, every lattice vector in the complex span of v lies in the \mathcal{R} -span of v . (This uses the fact that \mathcal{R} is a principal ideal domain.) Furthermore, by the selfduality of M , there exists $w \in M$ satisfying $\langle v|w \rangle = 1$. Then $R(w) = w - v(1 - \alpha)/v^2$ and so $w - R(w) = v(1 - \alpha)/v^2$ lies in M . Therefore $(1 - \alpha)/v^2 \in \mathcal{R}$. Unless $\alpha = -1$ this requires $v^2 = \pm 1$ and if $\alpha = -1$ then it requires that v^2 divide 2. \square

In order to compare our groups to the DM groups we will also need to consider the transformations of projective space that arise from linear reflections, which we call projective reflections. If L is a Lorentzian lattice then $P\text{Aut } L$ may contain projective reflections that are not represented by any reflection of L . For an example, consider $\text{Aut } II_{1,1}^{\mathcal{G}}$. The subgroup of elements of determinant one is conjugate to $\text{SL}_2 \mathbb{Z}$ and hence contains an element acting on $\mathbb{C}H^1$ as a triflection. This happens despite the fact (Lemma 8.1) that the only reflections of $II_{1,1}^{\mathcal{G}}$ are biflections. The following lemma assures us that this is merely a low-dimensional phenomenon.

Lemma 8.2. *Suppose M is an n -dimensional lattice over $\mathcal{R} = \mathcal{E}$ or \mathcal{G} and that R is a projective reflection in $P\text{Aut } M$, of order $m < n$. Then R is represented by a reflection of M .*

Proof: We will also write R for any element of $\text{Aut } M$ representing R . Since R acts on $\mathbb{C}P^{n-1}$ as a projective reflection, it has two distinct eigenvalues λ and λ' , with one (say λ) having multiplicity $n - 1$. Furthermore, since R^m preserves M and acts trivially on $\mathbb{C}P^{n-1}$, we see that there is a unit α of \mathcal{R} such that $\lambda^m = \lambda'^m = \alpha$. The characteristic polynomial of R is $(\lambda - x)^{n-1}(\lambda' - x)$, and since $R \in \text{GL}_n \mathcal{R}$ the coefficients must all lie in \mathcal{R} . We write y and z for the coefficients of x^{n-1} and x^{n-m-1} , and compute

$$\begin{aligned} \binom{n-1}{1} \lambda + \binom{n-1}{0} \lambda' &= (-1)^{n-1} y \\ \binom{n-1}{m+1} \lambda^{m+1} + \binom{n-1}{m} \lambda' \lambda^m &= (-1)^{n-m-1} z. \end{aligned}$$

Because $\lambda^m = \alpha \in \mathcal{R}$, the second equation reduces to a linear equation in λ and λ' . For $n > m$ this is a nonsingular system of equations, so $\lambda, \lambda' \in \mathcal{R} \otimes \mathbb{Q}$. Since λ, λ' are roots of unity they must actually lie in \mathcal{R} . Then $\lambda^{-1} R \in \text{Aut } M$ has eigenvalues 1 (with multiplicity $n - 1$) and $\lambda^{-1} \lambda'$, completing the proof. \square

Now we will consider the DM groups. If Γ is a group acting on $\mathbb{C}H^n$ then a projective reflection in Γ is called primitive if it is not a power of a projective reflection in Γ of larger order. The construction of the DM groups allows one to find primitive projective reflections in them. This requires a sketch of the construction, for which we use Thurston's approach. Let $n \geq 4$ and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n -tuple of numbers in the interval $(0, 2\pi)$ that sum to 4π . Let $P(\alpha)$ be the moduli space of pairs (p, g) where p is an injective map from $\{1, \dots, n\}$ to an oriented sphere S^2 and g is a singular Riemannian metric on S^2 which is flat except on the image of p , with $p(i)$ being a cone point of curvature α_i . We denote $p(i)$ also by p_i . Two such pairs are considered equivalent if they differ by an orientation-preserving similarity that identifies the corresponding points p_i with each other. This moduli space is a manifold of $2(n - 3)$ real dimensions and admits a metric which is locally isometric to $\mathbb{C}H^{n-3}$. Let H be the group of elements σ of the symmetric group S_n satisfying $\alpha_{\sigma(i)} = \alpha_i$ for all $i = 1, \dots, n$. Then H acts by isometries of $P(\alpha)$, by permuting the points p_i . We denote the quotient orbifold by $C(\alpha)$. The fundamental group of $P(\alpha)$ is the pure (spherical) braid group on n strands, and the orbifold fundamental group of $C(\alpha)$ is the subgroup of the full (spherical) braid group that maps to H under the usual map from the braid group to the symmetric group.

If the α_i satisfy certain conditions then the metric completion $\bar{C}(\alpha)$ of $C(\alpha)$ turns out to be the quotient of CH^{n-3} by a reflection group $\Gamma(\alpha)$. There are only 94 choices for α (with $n \geq 5$) satisfying these conditions, and the corresponding $\Gamma(\alpha)$ are the DM groups. The points of $\bar{C}(\alpha) \setminus C(\alpha)$ are the images of the mirrors of certain reflections of $\Gamma(\alpha)$. One can figure out the orders of the primitive reflections associated to these mirrors by finding the “cone angle” at each generic point of $\bar{C}(\alpha) \setminus C(\alpha)$: if the cone angle is $2\pi/m$ then the corresponding primitive projective reflections have order m . (This cone angle should not be confused with the cone angles at the points $p_i \in S^2$.) The generic points of $\bar{C}(\alpha) \setminus C(\alpha)$ are associated to “collisions” between pairs of points p_i and p_j on S^2 for which $\alpha_i + \alpha_j < 2\pi$. We quote Thurston’s Proposition 3.5, which provides a way to compute the cone angles at these points of $\bar{C}(\alpha) \setminus C(\alpha)$.

Proposition 8.3. *Let S be the stratum of $\bar{C}(\alpha_1, \dots, \alpha_n)$ where two cone points of S^2 of curvature α_i and α_j collide. If $\alpha_i = \alpha_j$ then the cone angle around S is $\pi - \alpha_i$; otherwise it is $2\pi - \alpha_i - \alpha_j$. \square*

For example, take α to be the 10-tuple $(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$, which is number 13 on Thurston’s list and number 4 on Mostow’s. Then at the singular strata of $\bar{C}(\alpha)$ where two cone points of curvature $\frac{2\pi}{3}$ (resp. two of curvature $\frac{\pi}{3}$, resp. one of each curvature) collide, the cone angle is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$ (resp. $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$, resp. $2\pi - \frac{2\pi}{3} - \frac{\pi}{3} = \pi$). We deduce that $\Gamma(\alpha)$ contains primitive projective reflections of orders 6, 3 and 2.

Theorem 8.4. *If L is $I_{n,1}^\mathcal{E}$ ($n \geq 4$) or $II_{4n+1,1}^S$ ($n \geq 1$) then $\text{Reflec } L$ does not appear among the Deligne-Mostow groups.*

Proof: By Lemmas 8.1 and 8.2, $\text{PAut } L$ contains no primitive projective reflections of order 3 or 4. Also, $\text{Aut } L$ is not cocompact because L contains isotropic vectors. Turning to the DM groups, Prop. 8.3 and the list of n -tuples α provided in [23] or [24] make it easy to compute the cone angles at all the generic points of $\bar{C}(\alpha) \setminus C(\alpha)$ for each n -tuple α with $n \geq 7$. The author wrote a short computer program to do this, and also performed the computation by hand. The only one for which none of the cone angles are $2\pi/4$ or $2\pi/3$ is number 50 on Thurston’s list (number 21 on Mostow’s). According to Thurston’s table, $\Gamma(\alpha)$ is cocompact for this choice of α . Therefore each DM groups acting on $\mathbb{C}H^n$ for $n \geq 4$ is either cocompact or contains a primitive projective reflection of order 3 or 4. The theorem follows. \square

We close by sketching a proof that $\text{Reflec } I_{3,1}^\mathcal{E}$ is one of the DM groups—it is the group $\Gamma(\alpha)$ with $\alpha = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$, which is number 1 on Thurston’s list and number 23 on Mostow’s. Because all the α_i are equal, the orbifold fundamental group of $C(\alpha)$ is the spherical braid group B_6 on six strands. A standard generator for B_6 , braiding two points p_i and p_{i+1} , corresponds to a loop in $C(\alpha)$ encircling the singular stratum S of $\bar{C}(\alpha)$ associated to a collision between p_i and p_{i+1} . Since the cone angle at S is $\pi/3$ we find that the standard generators map to 6-fold reflections. This fact, together with the braid relations and the fact that the image of B_6 is not finite, specifies the representation uniquely up to complex conjugation. The five standard generators may be taken to map to $(-\omega)$ -reflections in short roots of $I_{3,1}^\mathcal{E}$, which are orthogonal if the corresponding braid generators commute and have inner product +1 otherwise. One may then use the techniques of Sections 5 and 6 to show that the image of B_6 is $\text{Reflec}_0 I_{3,1}^\mathcal{E}$. The arguments we have sketched here concerning the braid group representation are carried out in detail in [2].

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